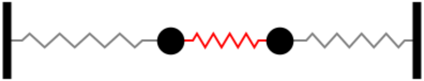
The $\lambda - \tau$ Inverse Eigenvalue Problem

Keivan Hassani Monfared
Joint work with Bryan Shader
University of Wyoming



Animation courtesy of Dr. Dan Russell, Grad. Prog. Acoustics, Penn State

Graph of a matrix

$A_{n \times n}$: real symmetric matrix
$G(A)$: a graph G on n vertices $1, 2, \ldots, n$
$i \sim j$ if and only if $i \neq j$ and $a_{ij} \neq 0$

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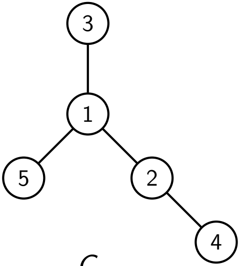

G

$$A = \begin{bmatrix} 2 & 1 & 3 & 0 & -4 \\ 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Then we say $A \in S(G)$.

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Given real numbers

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n$$

and a family F of matrices, does there exist a matrix $A \in F$ with eigenvalues λ_i's such that $A(1)$ has eigenvalues μ_i's?

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- connected graph [M, Shader 2013]

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Sketch of the proof for connected graphs:

- Choose a spanning tree of G, call it T.
- Solve the problem for T using Duarte's method, call it A.
- Show that the A is "generic", using a property similar to the Strong-Arnold Property.
- Perturb the zero entries, and the implicit function theorem guarantees the existence of a perturbation of the nonzero entries such that the eigenvalues of A and $A(1)$ remain the same, without zeroing out those zero entries.

How to show the Jacobian is nonsingular

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\operatorname{Jac}(F)\Big|_A = 2 * \left[\begin{array}{ccc|ccc}
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Direct calculation: $[A, X](i) = O$, so $p(A) = -\widetilde{q(B)}$, and $Ap(A) = p(A)\tilde{B}$, i.e. A, B are intertwining matrices, and either $p(A) = O$ or $A, \tilde{B}$ share an eigenvalue. But A, B do not share any eigenvalues, so the only possible case is that A has a zero eigenvalue. But it can be shown that in this case B also has a zero eigenvalue, which is a contradiction.

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So, $p(A) = O$, but $deg(min_A(x)) = n$ and $deg(p(x)) = n - 1$. i.e. $p(x), q(x)$ are zero polynomials, thus $\alpha = O$.

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Lemma:
Let A have the Duarte property with respect to the vertex i, $G(A)$ be a tree T, and X be a symmetric matrix such that

1. $I \circ X = O$,
2. $A \circ X = O$,
3. $[A, X](i) = O$.

then $X = O$.

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$$\lambda_i \leq \tau_i \leq \lambda_{i+2}$$

- From now on assume that a list of n λ's and $n-2$ τ's is given.
- We assume

$$\lambda_i < \tau_i < \lambda_{i+2}$$

$$\tau_i \neq \lambda_{i+1}$$

$$\begin{matrix} \lambda_1 & & & \lambda_2<\lambda_3 & & & \lambda_4 & & & \lambda_5<\lambda_6 & & & \lambda_7<\lambda_8 \\ & < & < & & < & < & & < & < & & < & < & \\ & \tau_1 & & & \tau_2 & & & \tau_3<\tau_4 & & & \tau_5<\tau_6 & & \end{matrix}$$

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- at most two τ's are consecutive, and we call them $\tau-$pairings.

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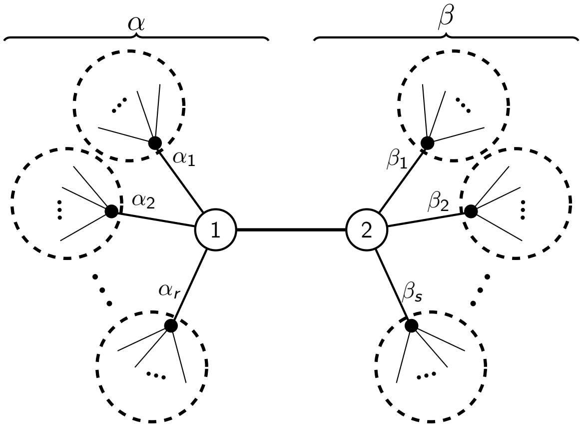
Proof: Counting, Cauchy interlacing inequalities, and pigeonhole principle.

The λ, τ problem for a tree:



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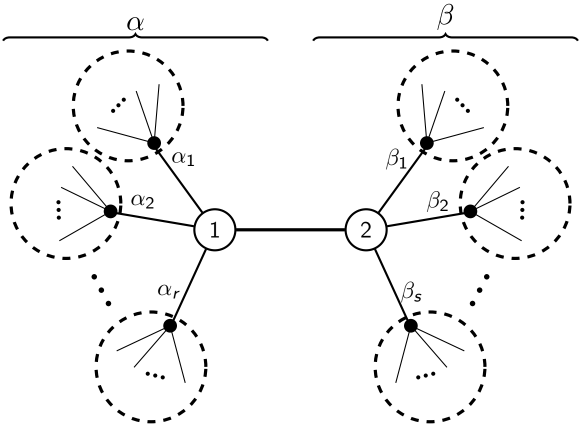

Lemma:

If λ_i's are eigenvalues of A and τ_i's are eigenvalues of $A(\{1,2\})$, then

$$\frac{\prod_{i=1}^{n}(x-\lambda_i)}{\prod_{i=1}^{n-2}(x-\tau_i)} = -a_{12}^2 + \frac{c_{A[\alpha]}}{c_{A[\alpha\setminus\{1\}]}}\frac{c_{A[\beta]}}{c_{A[\beta\setminus\{2\}]}}.$$

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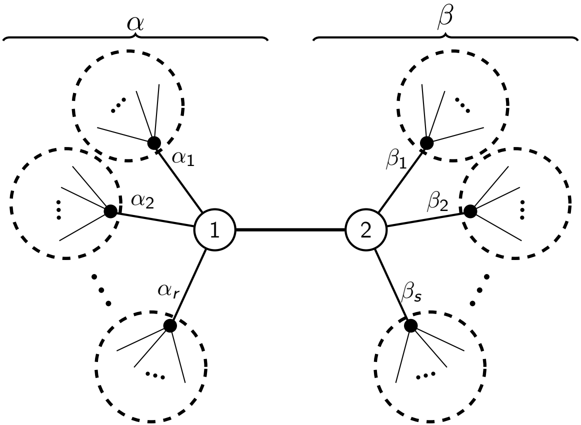

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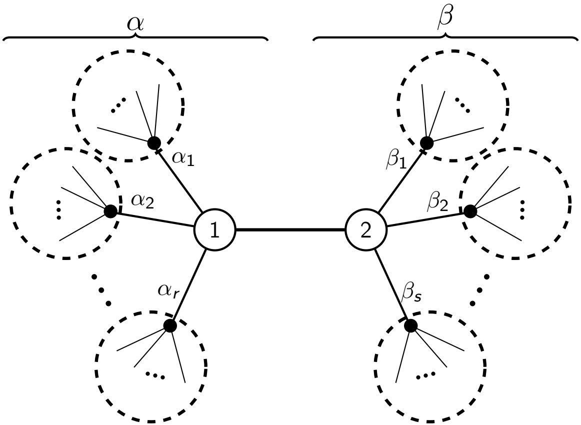

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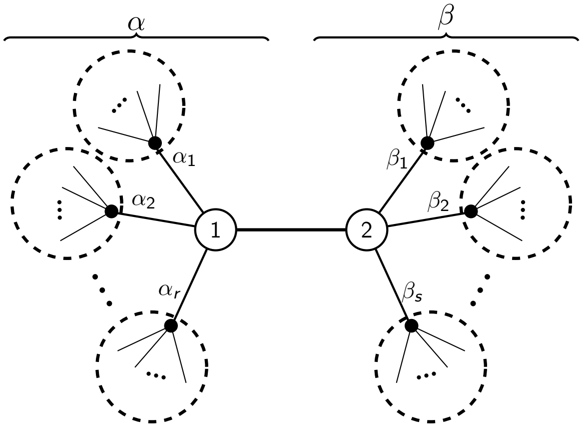

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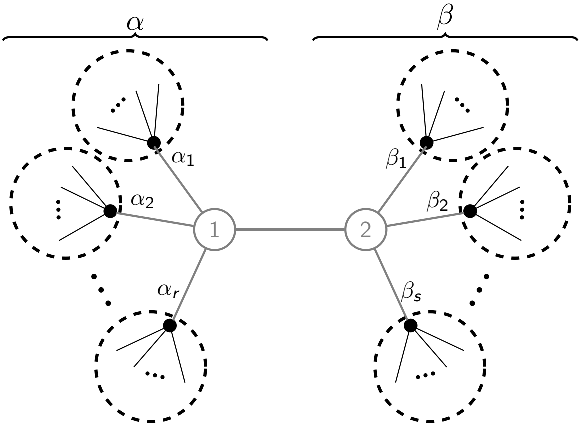

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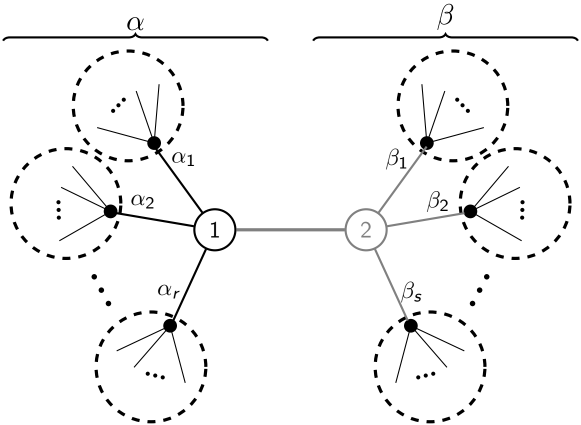

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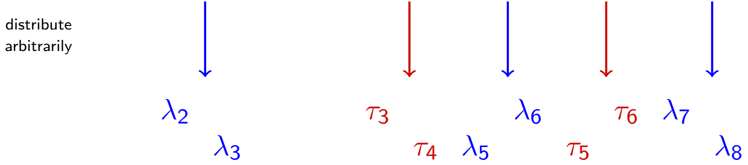

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$r = 9$: λ_2 τ_3 λ_6 τ_6 λ_7

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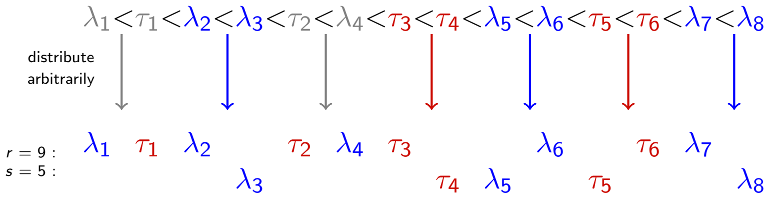

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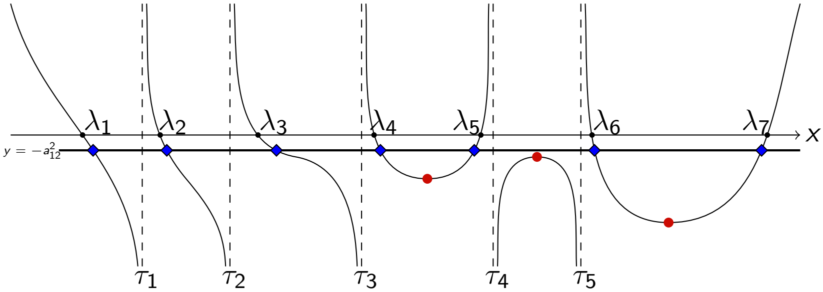

$$f(x) = \frac{\prod_{i=1}^{n}(x - \lambda_i)}{\prod_{i=1}^{n-2}(x - \tau_i)}.$$

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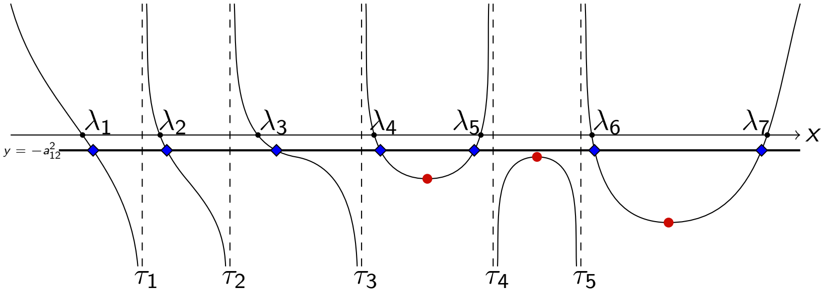

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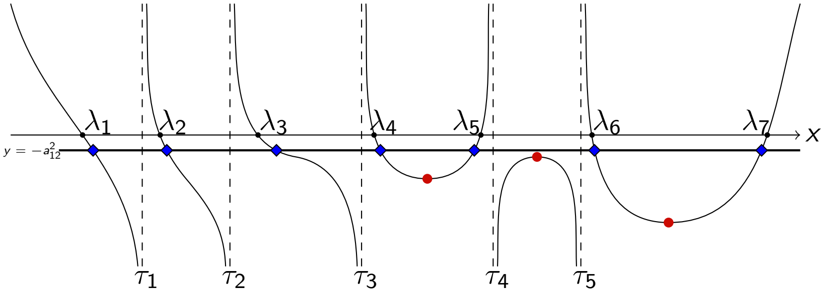

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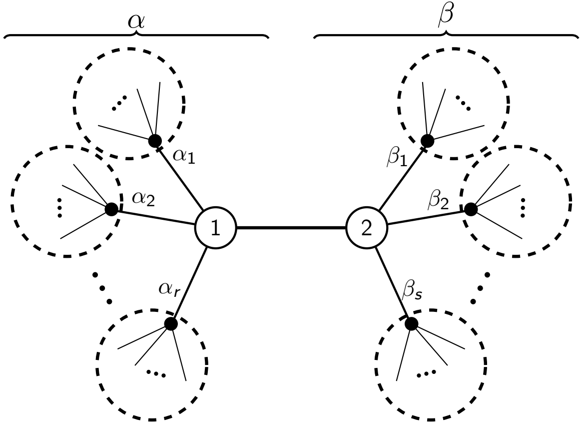
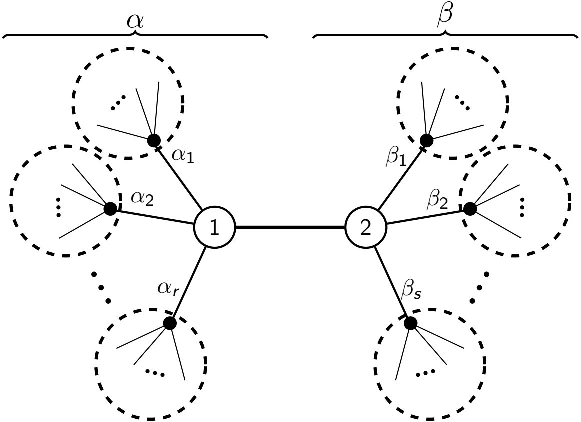
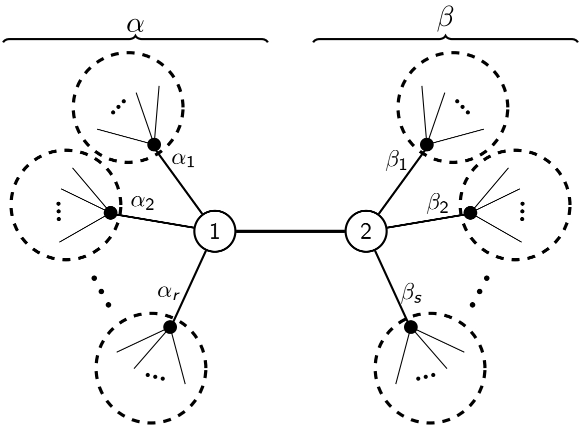
- Let blue diamonds to be μ's
- Distribute the list of μ's and τ's into two lists of sizes $2|\alpha| - 1$ and $2|\beta| - 1$



α
β
α1
α2
αr
β1
β2
βs
1
2



- Use Duarte's result to realize matrices for $A[\alpha]$ and $A[\beta]$.



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- Define A to be

$$\left(\begin{array}{c|c} A[\alpha] & a_{12} \\ \hline a_{12} & A[\beta] \end{array}\right)$$

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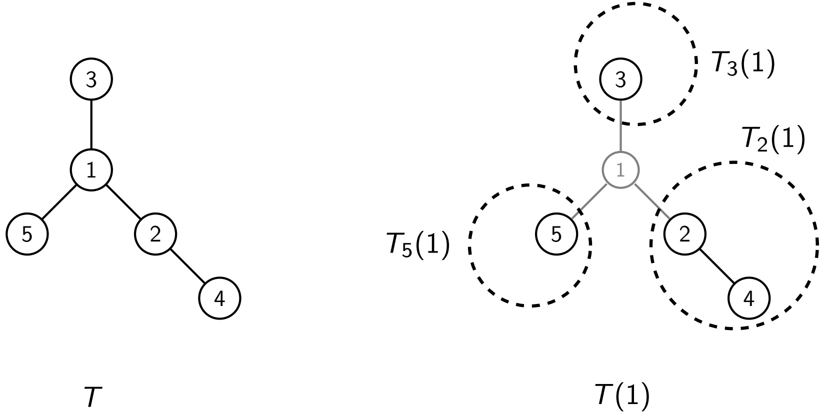
- Perturb the zero entries small enough, and the implicit function theorem guarantees some perturbation in nonzero entries keep the eigenvalues of A and $A(\{1,2\})$ fixed, without zeroing out those entries.

Thank you!

Trees and the Duarte Property



Matrix A has the Duarte property w.r.t to 1, when $A \in S(T)$

- Eigenvalues of $A(1)$ strictly interlace those of A,
- $A_2(1), A_3(1)$, and $A_5(1)$ have the Duarte property, w.r.t. 2,3, and 5, respectively.