The $\lambda - \mu$ Inverse Eigenvalue Problem

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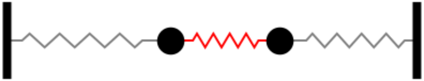
Joint work with Bryan Shader
University of Wyoming



Animation courtesy of Dr. Dan Russell, Grad. Prog. Acoustics, Penn State

Graph of a matrix

$A_{n\times n}$: real symmetric matrix
$G(A)$: a graph G on n vertices $1, 2, \ldots, n$
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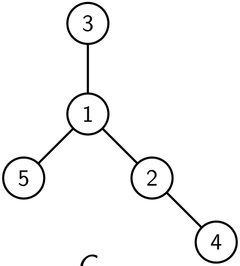

G

$$A = \begin{bmatrix} 2 & 1 & 3 & 0 & -4 \\ 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Then we say $A \in S(G)$.

The λ, μ problem:

Given real numbers

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n$$

and a family F of matrices, does there exist a matrix $A \in F$ with eigenvalues λ_i's such that $A(1)$ has eigenvalues μ_i's?

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- star [Fan, Pall 1957]
- path [Gladwell 1988]
- tree [Duarte 1989]

Theorem [Duarte 89]:

T : a **tree** with vertices 1,2, . . . , n
i : a vertex of T
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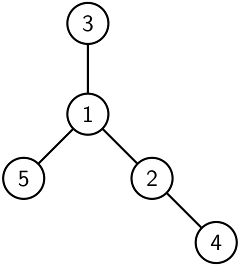

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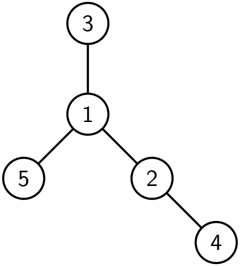

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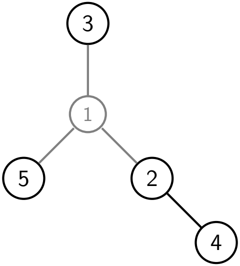

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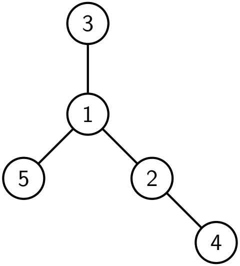

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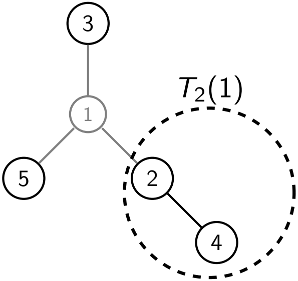

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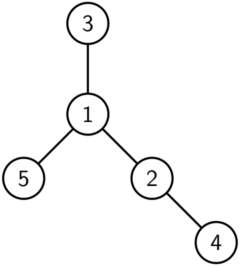

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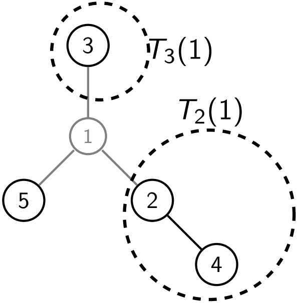

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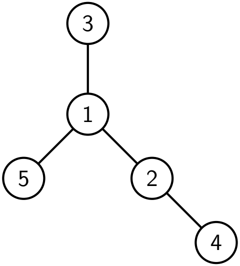

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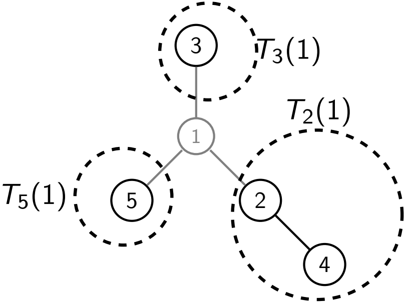

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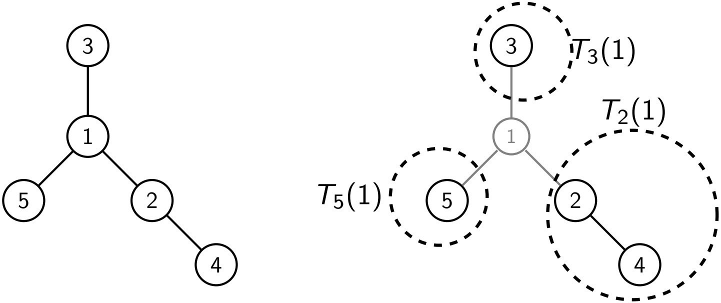


T $\qquad$ $T(1)$

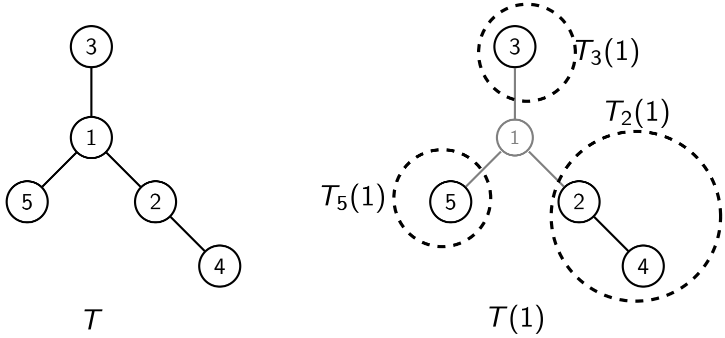
We say a matrix A where its graph is a tree T has the Duarte property w.r.t. vertex i if
either

- A is 1×1

or

- the eigenvalues of $A_j(i)$ strictly interlace those of A
- and $A_j(i)$ has the Duarte property w.r.t. vertex j

for all neighbours j of i.



Proof is by induction on the number of vertices.

$$\frac{f(\lambda)}{g(\lambda)} = (\lambda - a_{ii}) - \sum_{j=1}^{m} a_{ij}^2 \frac{h_j(\lambda)}{g_j(\lambda)}$$

- g_j: characteristic polynomial of $A_j(i)$
- a_{ij}: real number
- h_j: monic polynomial with $deg(h_j) < deg(g_j)$
- roots of h_j strictly interlace the roots of g_j

$$
A \xrightarrow{i^{th}\text{row}} \overset{\qquad\qquad\qquad\qquad i^{th}\text{ col}\downarrow}{\left[\begin{array}{c|c|c|c|c}
\ddots & & & & \iddots \\ \hline
 & B_{j_1} & a_{ij_1} & O & \\ \hline
 & a_{ij_1} & a_{ii} & a_{ij_2} & \\ \hline
 & O & a_{ij_2} & B_{j_2} & \\ \hline
\iddots & & & & \ddots
\end{array}\right]}
$$

- A realizes the given spectral data.

Theorem [M., Shader 13]:

G : a **connected** graph with vertices $1, 2, \ldots, n$
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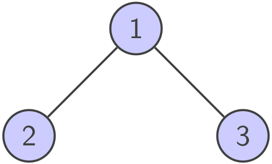
- Perturb the zero entries, and the implicit function theorem guarantees the existence of a perturbation of the nonzero entries such that the eigenvalues of A and $A(1)$ remain the same, without zeroing out those zero entries.

Example



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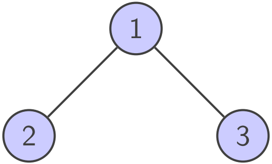

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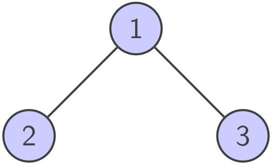

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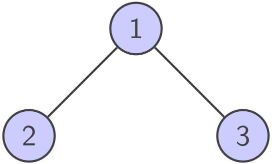

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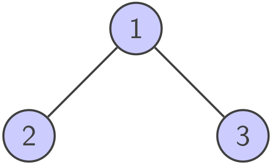

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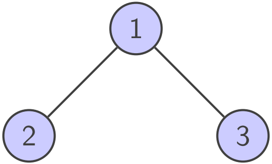

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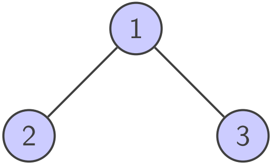

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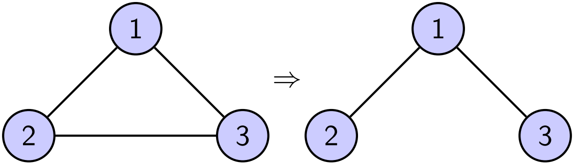
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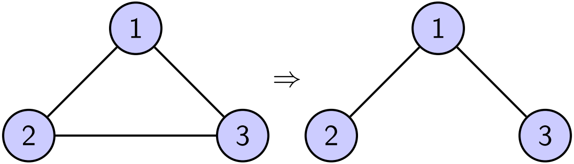

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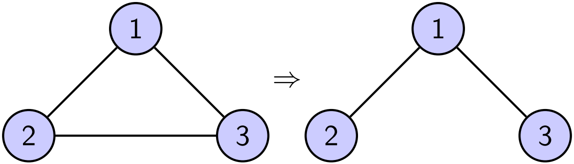

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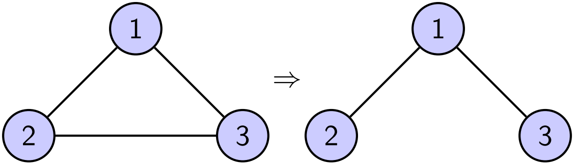

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Then

$$\begin{aligned} F(x_1, \ldots, x_5) =& (2(x_1 + x_2 + x_3), 4\,{x_1}^2 + 2\,{x_4}^2 + 2\,{x_5}^2 + 4\,{x_2}^2 + 4\,{x_3}^2, \\ & 8\,{x_1}^3 + 6\,x_1{x_4}^2 + 6\,x_1{x_5}^2 + 6\,{x_4}^2x_2 + 6\,{x_5}^2x_3 + 8\,{x_2}^3 + 8\,{x_3}^3, \\ & 2(x_2 + x_3), 4\,({x_2}^2 + x_3^2)) \end{aligned}$$

$$\mathrm{Jac}(F) = \left[\begin{array}{ccc|cc} 2 & 2 & 2 & 0 & 0 \\ 8x_1 & 8x_2 & 8x_3 & 4x_4 & 4x_5 \\ 24x_1^2 + 6x_4^2 + 6x_5^2 & 24x_2^2 + 6x_4^2 + 6x_5^2 & 24x_3^2 + 6x_5^2 & 12x_1x_4 + 12x_2x_4 & 12x_1x_5 + 12x_3x_5 \\ \hline 0 & 2 & 2 & 0 & 0 \\ 0 & 8x_2 & 8x_3 & 0 & 0 \end{array}\right]$$

$$\mathrm{Jac}(F) = \left[\begin{array}{ccc|cc} 2 & 2 & 2 & 0 & 0 \\ 8x_1 & 8x_2 & 8x_3 & 4x_4 & 4x_5 \\ 24x_1^2 + 6x_4^2 + 6x_5^2 & 24x_2^2 + 6x_4^2 + 6x_5^2 & 24x_3^2 + 6x_5^2 & 12x_1x_4 + 12x_2x_4 & 12x_1x_5 + 12x_3x_5 \\ \hline 0 & 2 & 2 & 0 & 0 \\ 0 & 8x_2 & 8x_3 & 0 & 0 \end{array}\right]$$

$$\det(\mathrm{Jac}(f)) = 1536\, x_4x_5{x_3}^2 - 3072\, x_4x_5x_3x_2 + 1536\, x_5x_4{x_2}^2$$

$$\mathrm{Jac}(F) = \left[\begin{array}{ccc|cc} 2 & 2 & 2 & 0 & 0 \\ 8x_1 & 8x_2 & 8x_3 & 4x_4 & 4x_5 \\ 24x_1^2+6x_4^2+6x_5^2 & 24x_2^2+6x_4^2+6x_5^2 & 24x_3^2+6x_5^2 & 12x_1x_4+12x_2x_4 & 12x_1x_5+12x_3x_5 \\ \hline 0 & 2 & 2 & 0 & 0 \\ 0 & 8x_2 & 8x_3 & 0 & 0 \end{array}\right]$$

$$\det(\mathrm{Jac}(f)) = 1536\, x_4 x_5 {x_3}^2 - 3072\, x_4 x_5 x_3 x_2 + 1536\, x_5 x_4 {x_2}^2$$

$$\det(\mathrm{Jac}(f)\Big|_A) = 4608\sqrt{132}$$

Let $y_1 = \frac{\sqrt{3}}{2}$, then

$$\widehat{M} = \begin{bmatrix} -\mathbf{8} & \frac{9+\sqrt{11}}{2\sqrt{2}} & \frac{\sqrt{66}-3\sqrt{6}}{4} \\ \frac{9+\sqrt{11}}{2\sqrt{2}} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{66}-3\sqrt{6}}{4} & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \xrightarrow{\text{eigenvalues}} -10, 0, 2$$

$$\widehat{N} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \xrightarrow{\text{eigenvalues}} -1, 1$$

Or let $y_1 = 0.1$, then

$$\widehat{M} \approx \begin{bmatrix} -\mathbf{8} & -3.552219778 & 2.526209542 \\ -3.552219778 & -0.9949874371 & 0.1 \\ 2.526209542 & 0.1 & 0.99498743710 \end{bmatrix}$$

$$\xrightarrow{\text{eigenvalues}} -9.999999999, -1.342005956 \cdot 10^{-15}, 1.999999999$$

$$\widehat{N} \approx \begin{bmatrix} -0.9949874371 & 0.1 \\ 0.1 & 0.99498743710 \end{bmatrix} \xrightarrow{\text{eigenvalues}} -1, 1$$

What does "generic" mean?

- Let $x := (x_1, \ldots, x_{2n-1}), y := (y_1, \ldots, y_{m-n+1})$
- $g : \mathbb{R}^{m+n} \to \mathbb{R}^{2n-1}$

$$g(x, y) := (c_0, c_1, \ldots, c_{n-1}, d_0, d_1, \ldots, d_{n-2})$$

  c_i : nonleading coeff's of the characteristic polynomial of M

  d_i : nonleading coeff's of the characteristic polynomial of N

- $f(x, y) := \left(\operatorname{tr} M, \operatorname{tr} M^2, \ldots, \operatorname{tr} M^n, \operatorname{tr} N, \operatorname{tr} N^2, \ldots, \operatorname{tr} N^{n-1}\right)$
- Newton's identities imply f is obtained from g by an invertible change of variables, i.e. $\operatorname{Jac}(g)\Big|_A$ is nonsingular iff $\operatorname{Jac}(f)\Big|_A$ is nonsingular
- $F(x) := f(x, 0)$. Then $\operatorname{Jac}(f)\Big|_A$ is nonsingular if $\operatorname{Jac}(F)\Big|_A$
- (Implicit Function Theorem) x_i's can be described as continuous functions of y_i's in a neighbourhood of A
- so changing each y_i to some ϵ_i one can find $\hat{x}_i$ such that

$$g(\hat{x}_1, \ldots, \hat{x}_{2n-1}, \epsilon_1 \ldots, \epsilon_{m-n+1}) = (c_0, \ldots, c_{n-1}, d_0, \ldots, d_{n-2})$$

How to compute the Jacobian of f

Let (i, j) be a nonzero position of M with corresponding variable x_t. Then

- $\frac{\partial}{\partial x_t}\left(\operatorname{tr} M^k\right) = 2kM_{ij}^{k-1}$
- $\frac{\partial}{\partial x_t}\left(\operatorname{tr} N^k\right) = \begin{cases} 2kN_{ij}^{k-1} & ;\ \text{if } i, j \neq n \\ 0 & ;\ \text{o.w} \end{cases} = 2k\widetilde{N}_{ij}^{k-1}$

$$\operatorname{Jac}(F)\Big|_A = 2 * \left[\begin{array}{ccc|ccc} I_{i_1 j_1} & \cdots & I_{i_{n-1} j_{n-1}} & I_{11} & \cdots & I_{nn} \\ 2A_{i_1 j_1} & \cdots & 2A_{i_{n-1} j_{n-1}} & 2A_{11} & \cdots & 2A_{nn} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ nA_{i_1 j_1}^{n-1} & \cdots & nA_{i_{n-1} j_{n-1}}^{n-1} & nA_{11}^{n-1} & \cdots & nA_{nn}^{n-1} \\ \hline \widetilde{I}_{i_1 j_1} & \cdots & \widetilde{I}_{i_{n-1} j_{n-1}} & \widetilde{I}_{11} & \cdots & \widetilde{I}_{nn} \\ 2\widetilde{B}_{i_1 j_1} & \cdots & 2\widetilde{B}_{i_{n-1} j_{n-1}} & 2\widetilde{B}_{11} & \cdots & 2\widetilde{B}_{nn} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (n-1)\widetilde{B}_{i_1 j_1}^{n-2} & \cdots & (n-1)\widetilde{B}_{i_{n-1} j_{n-1}}^{n-2} & (n-1)\widetilde{B}_{11}^{n-2} & \cdots & (n-1)\widetilde{B}_{nn}^{n-2} \end{array}\right]$$

How to show the above matrix is nonsingular?

Lemma:

Let A have the Duarte property with respect to the vertex 1, $G(A)$ be a tree T, and X be a symmetric matrix such that

1. $I \circ X = O$,
2. $A \circ X = O$,
3. $[A, X](1) = O$.

then $X = O$.

How Does the Implicit Function Theorem Work?

Theorem

$x \in \mathbb{R}^s$, $y \in \mathbb{R}^r$

$F : \mathbb{R}^{s+r} \to \mathbb{R}^s$: *continuously differentiable on an open subset* U *of* $\mathbb{R}^{s+r}$

$$F(x, y) = (F_1(x, y), F_2(x, y), \dots, F_s(x, y)),$$

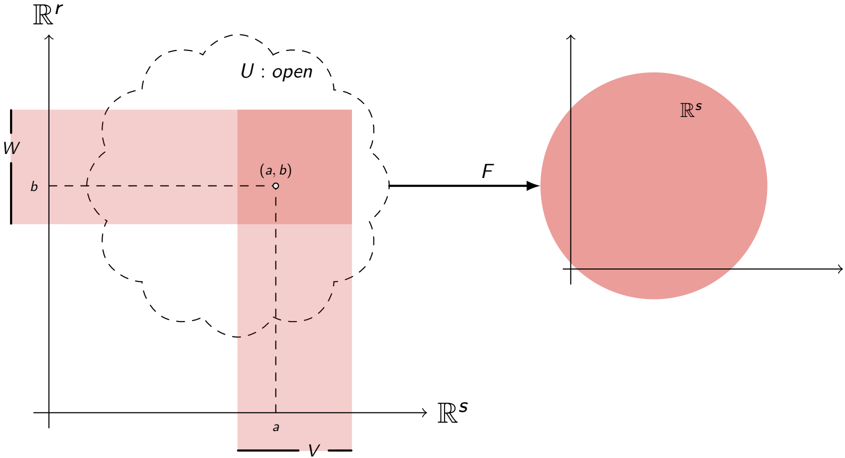
$(a, b) \in U$ *with* $a \in \mathbb{R}^s$, $b \in \mathbb{R}^r$

$c \in \mathbb{R}^s$ *such that* $F(a, b) = c$

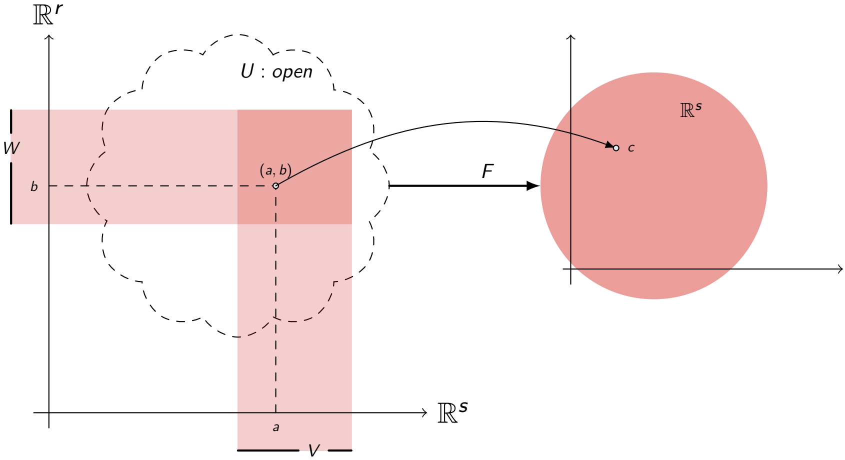
If $\left[\frac{\partial F_i}{\partial x_j} \Big|_{(a,b)} \right]$ *is nonsingular, then there exist an open neighborhood* V *containing* a *and an open neighborhood* W *containing* b *such that* $V \times W \subseteq U$ *and for each* $y \in W$ *there is an* $x \in V$ *with*

$$F(x, y) = c$$

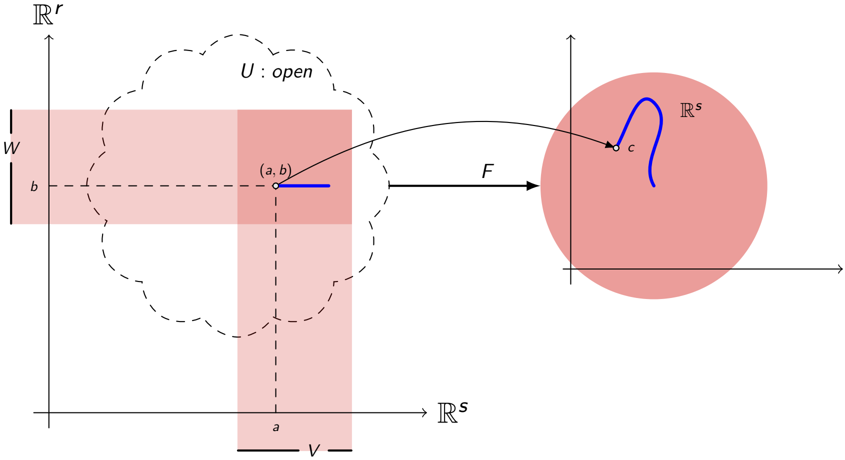
How Does the Implicit Function Theorem Work?



How Does the Implicit Function Theorem Work?



How Does the Implicit Function Theorem Work?



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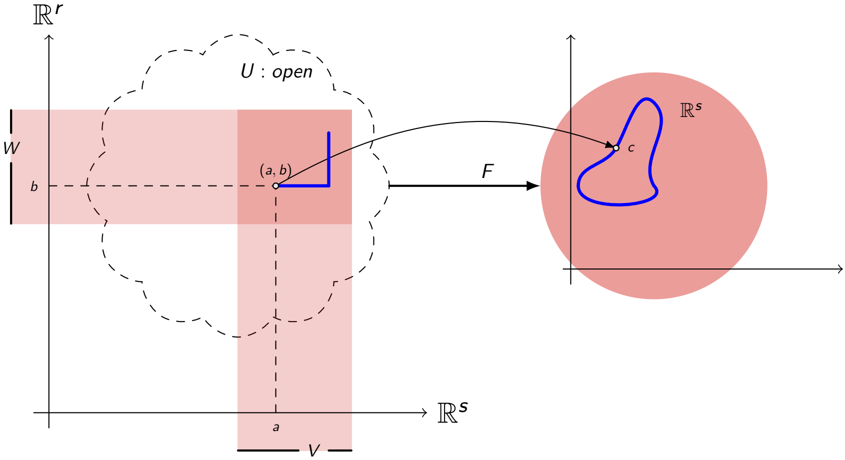
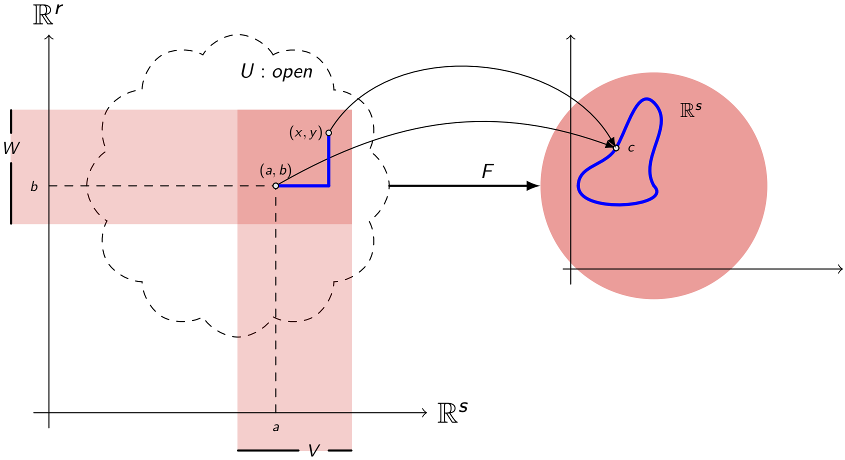
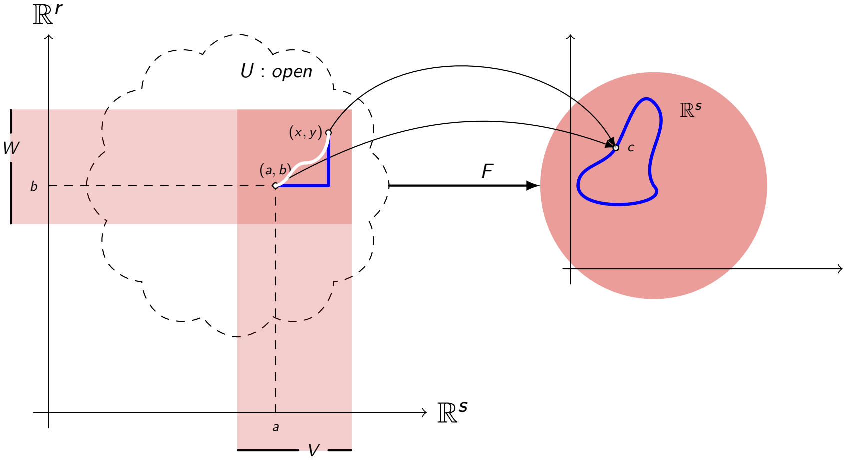


How Does the Implicit Function Theorem Work?



How Does the Implicit Function Theorem Work?



Thank You!!