Reconstruction of symmetric matrices with a given graph from interlaced spectral data

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Preliminary Exam

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Introduction

Combinatorial matrix theory

Spectral properties of a matrix (sub-matrix)

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$$\textbf{matrix} \quad \longleftrightarrow \quad \textbf{spectrum}$$

Combinatorial matrix theory

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Combinatorial matrix theory

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etc.

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$G(A)$: a graph G on n vertices $1, 2, \ldots, n$
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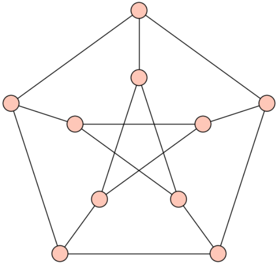
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & -5 & 0 & 0 & 0 & 0 \\ 1 & -2 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ -5 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$



Question

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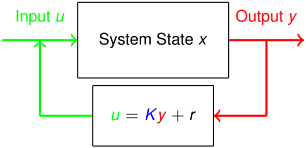
- **I**nverse **E**igenvalue **P**roblems
- IEP's appear in various engineering contexts

Applications

- control design
- system identification
- seismic tomography
- principal component analysis
- exploration and remote sensing
- antenna array processing
- geophysics
- molecular spectroscopy
- particle physics
- structure analysis
- circuit theory
- mechanical system simulation
- ...

Control System Design

A dynamic system:



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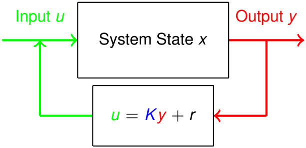

Feedback System:

$$\dot{x} = Ax + BKy + Br$$
$$= (A + BKC)x + Br$$
$$y = Cx$$

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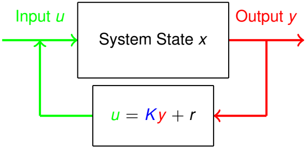

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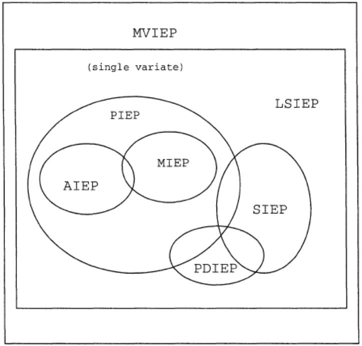
Objectives

Choose K to:

1. assign eigenvalues / stabilize (left half plane)
2. assign eigenvectors - inputs/outputs
3. ensure robustness (insensitivity to disturbances)

Classification of inverse eigenvalue problems, Chu 98

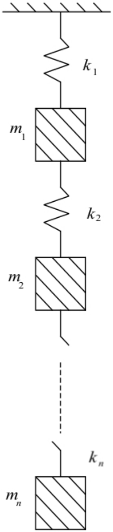


"Perhaps the most focused IEPs are structured problems, where a matrix with a specified structure as well as a designated spectrum is sought after. A lot of times this structure comes from the adjacency matrix of a graph."

Motivation

Example of a *fixed-free* system

k_i : Hooke's law constants
m_i : masses



Vibrations described with Newton's law of motion:

$$m_r \ddot{u}_r = F_r + \theta_{r+1} - \theta_r, \quad r = 1, 2, \ldots, n-1$$

$$m_n \ddot{u}_n = F_n - \theta_n$$

By Hooke's law:

$$\theta_r = k_r(u_r - u_{r-1}), \quad r = 1, 2, \ldots, n$$

$$u_0 = 0$$

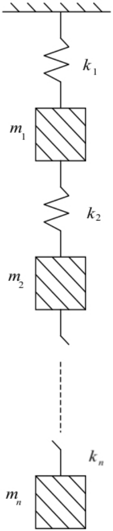
Altogether:

$$M\ddot{u} + Ku = F$$

Example of a *fixed-free* system

$$M\ddot{u} + Ku = F$$

$$K = \begin{bmatrix} k_0 + k_1 & -k_1 & & & \\ -k_1 & k_1 + k_2 & -k_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-2} & k_{n-2} + k_{n-1} & -k_{n_1} \\ & & & -k_{n-1} & k_{n-1} \end{bmatrix} = \text{stiffness}$$



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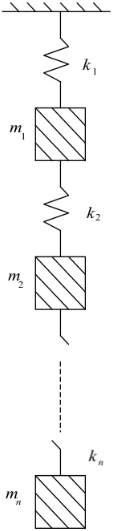

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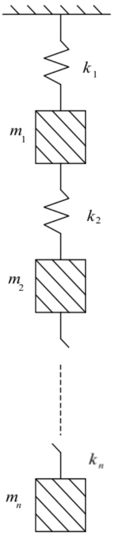

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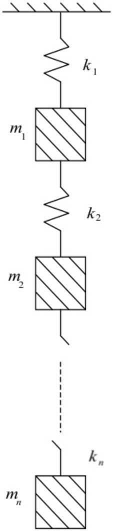

- fixed and free?
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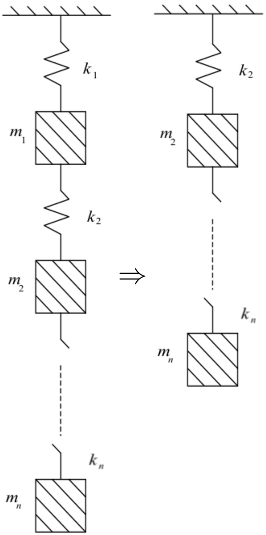
- IEP: Is there K such that λ_i are evalues of K?

Example of a *fixed-free* system

Jacobi matrix: a symmetric tridiagonal matrix K with negative off-diagonal entries.

IEP: Is there a Jacobi matrix such that eigenvalues of K are $\lambda_1, \ldots, \lambda_n$ and eigenvalues of $K(1)$ are $\mu_1, \ldots, \mu_{n-1}$?



Previous Results

Theorem (Gladwell 88)

For given $\{\lambda_i\}_{i=1}^{n}$ *and* $\{\mu_i\}_{i=1}^{n-1}$ *there is a Jacobi matrix* T *with*

$$\sigma(T) = \{\lambda_1, \ldots, \lambda_n\}$$

and

$$\sigma(T(j)) = \{\mu_1, \ldots, \mu_{n-1}\}$$

if and only if

$$\lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{n-1} < \lambda_n.$$

Moreover such Jacobi matrix is unique.

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- (New result) Any connected graph on n vertices realizes

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Previous Results

Theorem (Duarte 79)

*T : a **tree** with vertices 1,2, ..., n*
i : a vertex of T
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Then there is a (real) symmetric matrix A with graph T and eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $A(i)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$

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- For $k = 2$

$$A = \begin{bmatrix} \mu_1 & x \\ x & y \end{bmatrix}, y = \lambda_1 + \lambda_2 - \mu_1, x = \sqrt{(\lambda_2 - \mu_1)(\mu_1 - \lambda_1)}$$

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$$f(\lambda) := (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$
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- Partial Fraction Decomposition:
  - $\exists!$ a, and positive $y_1, y_1, \ldots, y_m$
  - $\exists!$ monic polynomials $h_1, h_2, \ldots, h_m$, with $deg(h_j) < deg(g_j)$ such that

$$\frac{f(\lambda)}{g(\lambda)} = (\lambda - a) - \sum_{j=1}^{m} y_j \frac{h_j(\lambda)}{g_j(\lambda)}$$

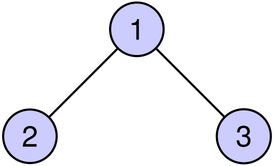
  - Furthermore, roots of h_j strictly interlace the roots of g_j

Previous Results

$$
\begin{array}{c}
 \hspace{8em} i^{th}\ \text{col} \downarrow \\
A \xrightarrow{i^{th}\text{row}} \left[\begin{array}{c|c|c|c|c}
\ddots & & & & \cdot^{\cdot^{\cdot}} \\ \hline
 & B_{j_1} & \sqrt{y_{j_1}} & O & \\ \hline
 & \sqrt{y_{j_1}} & a & \sqrt{y_{j_2}} & \\ \hline
 & O & \sqrt{y_{j_2}} & B_{j_2} & \\ \hline
\cdot^{\cdot^{\cdot}} & & & & \ddots
\end{array}\right]
\end{array}
$$

- One can show that A realizes the given spectral data.

Example



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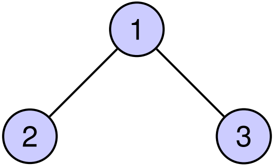

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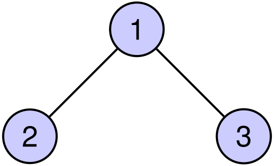

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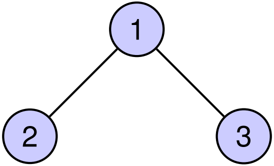

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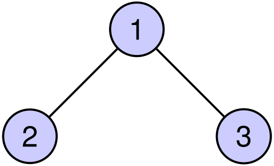

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$x_2 = -1, x_3 = 1$

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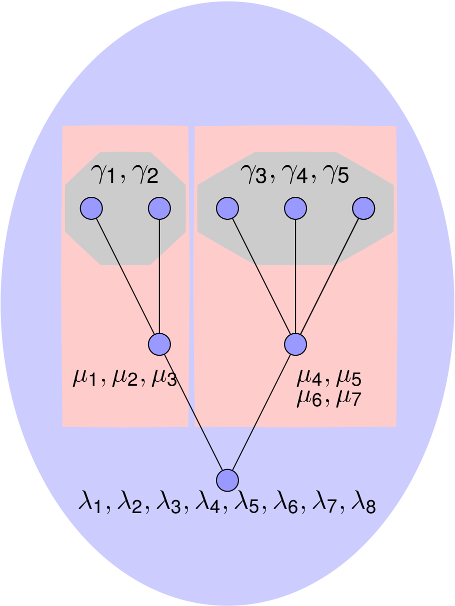
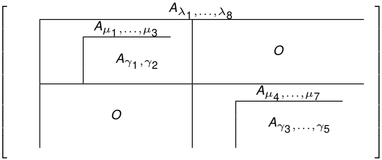
The big picture





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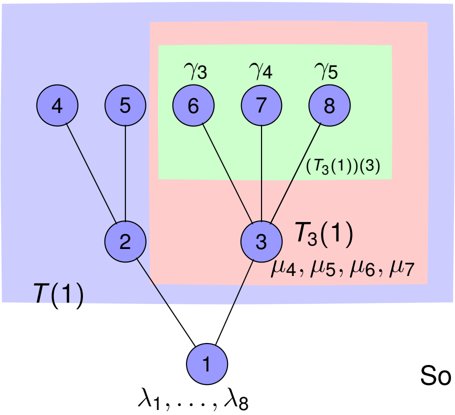
Proof:

- *G* is connected, then *G* has a spanning tree *T*
- (Duarte) *T* realizes λ's and μ's $\rightarrow A, B := A(i)$
- *A* has the Duarte property w.r.t to *i*, *that is A is "generic"*

The Duarte Property

Consider the tree T :



e-values of $A(1)$ strictly interlace

those of A, and $A_2(1)$ and $A_3(1)$

have the Duarte property

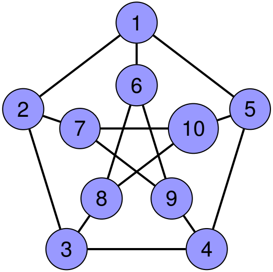
γs strictly interlace μs

μs strictly interlace λs

So A has the Duarte property w.r.t. 1

Main Result

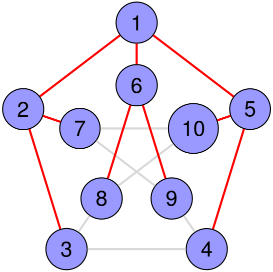
- Consider the following matrix with its graph



$$\begin{pmatrix}
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}$$

Main Result

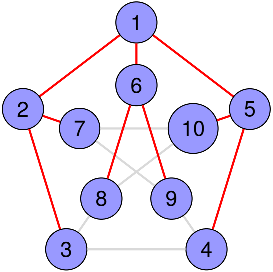
- Choose a spanning tree of the graph



$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Main Result

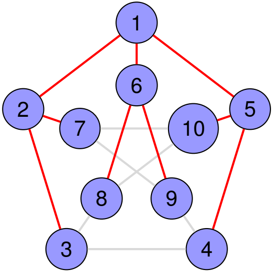
- Substitute the diagonal entries by $2x_k$, $k = 1, 2, \ldots, n$



$$\begin{pmatrix}
2x_1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 2x_2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 2x_3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 2x_4 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 2x_5 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 2x_6 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 2x_7 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 2x_8 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 2x_9 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 2x_{10}
\end{pmatrix}$$

Main Result

- Substitute the diagonal entries by x_k, $k = 1, 2, \ldots, n$ and nonzero entry of A at i_k, j_k by x_{n+k}, $k = 1, 2, \ldots, n-1$



$$\begin{pmatrix}
2x_1 & x_{11} & 0 & 0 & x_{12} & x_{13} & 0 & 0 & 0 & 0 \\
x_{11} & 2x_2 & x_{14} & 0 & 0 & 0 & x_{15} & 0 & 0 & 0 \\
0 & x_{14} & 2x_3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 2x_4 & x_{16} & 0 & 0 & 0 & 1 & 0 \\
x_{12} & 0 & 0 & x_{16} & 2x_5 & 0 & 0 & 0 & 0 & x_{17} \\
x_{13} & 0 & 0 & 0 & 0 & 2x_6 & 0 & x_{18} & x_{19} & 0 \\
0 & x_{15} & 0 & 0 & 0 & 0 & 2x_7 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & x_{18} & 0 & 2x_8 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & x_{19} & 1 & 0 & 2x_9 & 0 \\
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\end{pmatrix}$$

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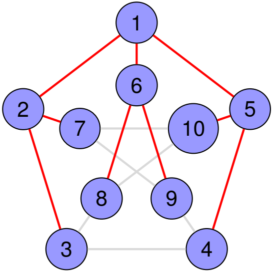
- Substitute the diagonal entries by x_k, $k = 1, 2, \ldots, n$ and nonzero entry of A at i_k, j_k by x_{n+k}, $k = 1, 2, \ldots, n-1$ and rest of the nonzero entries at i_k, j_k position of the matrix of G by y_k, $k = 1, 2, \ldots, m-n+1$ this gives M, and $N := M(i)$



$$\begin{pmatrix}
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x_{11} & 2x_2 & x_{14} & 0 & 0 & 0 & x_{15} & 0 & 0 & 0 \\
0 & x_{14} & 2x_3 & y_1 & 0 & 0 & 0 & y_2 & 0 & 0 \\
0 & 0 & y_1 & 2x_4 & x_{16} & 0 & 0 & 0 & y_3 & 0 \\
x_{12} & 0 & 0 & x_{16} & 2x_5 & 0 & 0 & 0 & 0 & x_{17} \\
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- $g : \mathbb{R}^{m+n} \to \mathbb{R}^{2n-1}$

$$g(x, y) := (c_0, c_1, \ldots, c_{n-1}, d_0, d_1, \ldots, d_{n-2})$$

c_i : nonleading coeff's of the characteristic polynomial of M

d_i : nonleading coeff's of the characteristic polynomial of N

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d_i : nonleading coeff's of the characteristic polynomial of N

- $f(x, y) := \left(\operatorname{tr} M, \operatorname{tr} M^2, \ldots, \operatorname{tr} M^n, \operatorname{tr} N, \operatorname{tr} N^2, \ldots, \operatorname{tr} N^{n-1}\right)$

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- $F(x) := f(x, 0)$. Then $\operatorname{Jac}(f)\Big|_A$ is nonsingular if $\operatorname{Jac}(F)\Big|_A$

Main Result

$$\mathrm{Jac}(F)\Big|_A = 2 * \left[\begin{array}{ccc|ccc} I_{i_1 j_1} & \cdots & I_{i_{n-1} j_{n-1}} & I_{11} & \cdots & I_{nn} \\ 2A_{i_1 j_1} & \cdots & 2A_{i_{n-1} j_{n-1}} & 2A_{11} & \cdots & 2A_{nn} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ nA^{n-1}_{i_1 j_1} & \cdots & nA^{n-1}_{i_{n-1} j_{n-1}} & nA^{n-1}_{11} & \cdots & nA^{n-1}_{nn} \\ \hline \widetilde{I}_{i_1 j_1} & \cdots & \widetilde{I}_{i_{n-1} j_{n-1}} & \widetilde{I}_{11} & \cdots & \widetilde{I}_{nn} \\ 2\widetilde{B}_{i_1 j_1} & \cdots & 2\widetilde{B}_{i_{n-1} j_{n-1}} & 2\widetilde{B}_{11} & \cdots & 2\widetilde{B}_{nn} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (n-1)\widetilde{B}^{n-2}_{i_1 j_1} & \cdots & (n-1)\widetilde{B}^{n-2}_{i_{n-1} j_{n-1}} & (n-1)\widetilde{B}^{n-2}_{11} & \cdots & (n-1)\widetilde{B}^{n-2}_{nn} \end{array}\right]$$

$$\tilde{B} = \left[\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right]$$

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- $\mathrm{Jac}(F)\Big|_A$ is nonsingular

Main Result

$$
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2A_{i_1 j_1} & \cdots & 2A_{i_{n-1} j_{n-1}} & 2A_{11} & \cdots & 2A_{nn} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
nA^{n-1}_{i_1 j_1} & \cdots & nA^{n-1}_{i_{n-1} j_{n-1}} & nA^{n-1}_{11} & \cdots & nA^{n-1}_{nn} \\
\hline
\widetilde{I}_{i_1 j_1} & \cdots & \widetilde{I}_{i_{n-1} j_{n-1}} & \widetilde{I}_{11} & \cdots & \widetilde{I}_{nn} \\
2\widetilde{B}_{i_1 j_1} & \cdots & 2\widetilde{B}_{i_{n-1} j_{n-1}} & 2\widetilde{B}_{11} & \cdots & 2\widetilde{B}_{nn} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(n-1)\widetilde{B}^{n-2}_{i_1 j_1} & \cdots & (n-1)\widetilde{B}^{n-2}_{i_{n-1} j_{n-1}} & (n-1)\widetilde{B}^{n-2}_{11} & \cdots & (n-1)\widetilde{B}^{n-2}_{nn}
\end{array}\right]
$$

$$
\tilde{B} = \left[\begin{array}{c|ccc}
0 & 0 & \cdots & 0 \\
\hline
0 & & & \\
\vdots & & B & \\
0 & & &
\end{array}\right]
$$

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$$\tilde{B} = \left[\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right]$$

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- $\left.\mathrm{Jac}(g)\right|_A$ is nonsingular

Main Result

- (Implicit Function Theorem) x_i's can be described as continuous functions of y_i's in a neighbourhood of A

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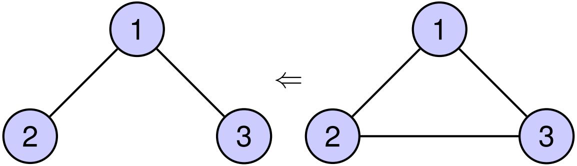
- (Implicit Function Theorem) x_i's can be described as continuous functions of y_i's in a neighbourhood of A
- so changing each y_i to some ϵ_i one can find $\hat{x}_i$ such that

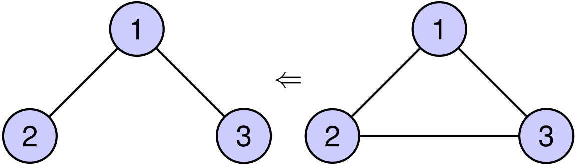
$$g(\hat{x}_1, \ldots, \hat{x}_{2n-1}, \epsilon_1 \ldots, \epsilon_{m-n+1}) = (c_0, \ldots, c_{n-1}, d_0, \ldots, d_{n-2})$$

□

Example

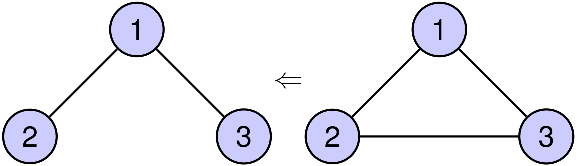


Example



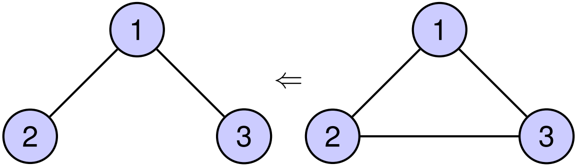
$$A = \begin{bmatrix} -8 & \sqrt{\frac{27}{2}} & \sqrt{\frac{11}{2}} \\ \sqrt{\frac{27}{2}} & -1 & 0 \\ \sqrt{\frac{11}{2}} & 0 & 1 \end{bmatrix}$$

Example



$$A = \begin{bmatrix} -8 & \sqrt{\frac{27}{2}} & \sqrt{\frac{11}{2}} \\ \sqrt{\frac{27}{2}} & -1 & 0 \\ \sqrt{\frac{11}{2}} & 0 & 1 \end{bmatrix} \Rightarrow M = \begin{bmatrix} 2x_1 & x_4 & x_5 \\ x_4 & 2x_2 & y_1 \\ x_5 & y_1 & 2x_3 \end{bmatrix}, N = \begin{bmatrix} 2x_2 & y_1 \\ y_1 & 2x_3 \end{bmatrix}$$

Example



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Then

$$\begin{aligned} F(x_1, \ldots, x_5, 0) = &(2(x_1 + x_2 + x_3), 4\,{x_1}^2 + 2\,{x_4}^2 + 2\,{x_5}^2 + 4\,{x_2}^2 + 4\,{x_3}^2, \\ &8\,{x_1}^3 + 6\,x_1{x_4}^2 + 6\,x_1{x_5}^2 + 6\,{x_4}^2x_2 + 6\,{x_5}^2x_3 + 8\,{x_2}^3 + 8\,{x_3}^3, \\ &2(x_2 + x_3), 4\,({x_2}^2 + x_3^2)) \end{aligned}$$

Example

$$\text{Jac}(F) = \left[\begin{array}{ccc|cc} 2 & 2 & 2 & 0 & 0 \\ 8x_1 & 8x_2 & 8x_3 & 4x_4 & 4x_5 \\ 24x_1^2 + 6x_4^2 + 6x_5^2 & 24x_2^2 + 6x_4^2 + 6x_5^2 & 24x_3^2 + 6x_5^2 & 12x_1x_4 + 12x_2x_4 & 12x_1x_5 + 12x_3x_5 \\ \hline 0 & 2 & 2 & 0 & 0 \\ 0 & 8x_2 & 8x_3 & 0 & 0 \end{array}\right]$$

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$$\det(\text{Jac}(f)) = 1536\, x_4 x_5 {x_3}^2 - 3072\, x_4 x_5 x_3 x_2 + 1536\, x_5 x_4 {x_2}^2 \rightarrow \text{(no } x_1\text{)}$$

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$$\det(\mathrm{Jac}(f)\Big|_A) = 4608\sqrt{132}$$

Example

Let $y_1 = \frac{\sqrt{3}}{2}$, then

$$\widehat{M} = \begin{bmatrix} \mathbf{-8} & \frac{9+\sqrt{11}}{2\sqrt{2}} & \frac{\sqrt{66}-3\sqrt{6}}{4} \\ \frac{9+\sqrt{11}}{2\sqrt{2}} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{66}-3\sqrt{6}}{4} & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \xrightarrow{\text{eigenvalues}} -10, 0, 2$$

$$\widehat{N} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \xrightarrow{\text{eigenvalues}} -1, 1$$

Example

Or let $y_1 = 0.1$, then

$$\widehat{M} \approx \begin{bmatrix} -\mathbf{8} & -3.552219778 & 2.526209542 \\ -3.552219778 & -0.9949874371 & 0.1 \\ 2.526209542 & 0.1 & 0.99498743710 \end{bmatrix}$$

$$\xrightarrow{\text{eigenvalues}} -9.999999999, -1.342005956 \cdot 10^{-15}, 1.999999999$$

$$\widehat{N} \approx \begin{bmatrix} -0.9949874371 & 0.1 \\ 0.1 & 0.99498743710 \end{bmatrix} \xrightarrow{\text{eigenvalues}} -1, 1$$

Some Additional Details

Newton's Identities

Let

$$p_k(x_1,\ldots,x_n) = \sum_{i=1}^{n} x_i^k = x_1^k + \cdots + x_n^k$$

and

$$\begin{aligned}
e_0(x_1,\ldots,x_n) &= 1,\\
e_1(x_1,\ldots,x_n) &= x_1 + x_2 + \cdots + x_n,\\
e_2(x_1,\ldots,x_n) &= \sum_{1\le i<j\le n} x_i x_j,\\
e_n(x_1,\ldots,x_n) &= x_1 x_2 \cdots x_n,\\
e_k(x_1,\ldots,x_n) &= 0, \quad \text{for } k > n.
\end{aligned}$$

Then

$$k e_k(x_1,\ldots,x_n) = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i}(x_1,\ldots,x_n) p_i(x_1,\ldots,x_n)$$

Newton's Identities

Consider the characteristic polynomial of A:

$$\prod_{i=1}^{n}(t - x_i) = \sum_{k=0}^{n}(-1)^k a_k t^{n-k}$$

and

$$s_k = p_k(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^k$$

Then

$$\begin{aligned}
s_1 &= a_1, \\
s_2 &= a_1 s_1 - 2a_2, \\
s_3 &= a_1 s_2 - a_2 s_1 + 3a_3, \\
s_4 &= a_1 s_3 - a_2 s_2 + a_3 s_1 - 4a_4, \\
&\vdots
\end{aligned}$$

and $s_k = \operatorname{tr} A^k$. So $\operatorname{tr} A, \ldots, \operatorname{tr} A^n$ uniquely determine the coefficients of the characteristic polynomial of A.

Jacobian

Lemma

Let (i,j) be a nonzero position of M with corresponding variable x_t. Then

(a) $\dfrac{\partial}{\partial x_t}\left(\operatorname{tr} M^k\right) = 2kM_{ij}^{k-1}$

(b) $\dfrac{\partial}{\partial x_t}\left(\operatorname{tr} N^k\right) = \begin{cases} 2kN_{ij}^{k-1} & ; \textit{if } i,j \neq n \\ 0 & ; \textit{o.w} \end{cases} = 2k\widehat{N}_{ij}^{k-1}$

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If $i \neq j$

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If $i = j$

$$\frac{\partial}{\partial x_t} M = 2E_{ii} = E_{ij} + E_{ji}.$$

Jacobian

in either case

$$\begin{aligned}
\frac{\partial}{\partial x_t}\left(\operatorname{tr}(M^k)\right) &= \sum_{l=0}^{k-1} \operatorname{tr}\left(M^l \cdot \frac{\partial}{\partial x_t} M \cdot M^{k-l-1}\right) && \text{(by the chain rule)} \\
&= \sum_{l=0}^{k-1} \operatorname{tr}\left(M^{k-1} \cdot \frac{\partial}{\partial t} M\right) && \text{(since } \operatorname{tr}(AB) = \operatorname{tr}(BA) \text{ for any } A \text{ and } B\text{)} \\
&= k \operatorname{tr}\left(M^{k-1}(E_{ij} + E_{ji})\right) \\
&= k\left((M^{k-1})_{ij} + (M^{k-1})_{ji}\right) \\
&= 2k(M^{k-1})_{ij}. && \text{(since } M \text{ is symmetric)}
\end{aligned}$$

Jacobian

Corollary

$$
Jac(F)\Big|_A = 2 * \left[\begin{array}{ccc|ccc}
I_{i_1 j_1} & \cdots & I_{i_{n-1} j_{n-1}} & I_{11} & \cdots & I_{nn} \\
2A_{i_1 j_1} & \cdots & 2A_{i_{n-1} j_{n-1}} & 2A_{11} & \cdots & 2A_{nn} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
nA^{n-1}_{i_1 j_1} & \cdots & nA^{n-1}_{i_{n-1} j_{n-1}} & nA^{n-1}_{11} & \cdots & nA^{n-1}_{nn} \\
\hline
\widetilde{I}_{i_1 j_1} & \cdots & \widetilde{I}_{i_{n-1} j_{n-1}} & \widetilde{I}_{11} & \cdots & \widetilde{I}_{nn} \\
2\widetilde{B}_{i_1 j_1} & \cdots & 2\widetilde{B}_{i_{n-1} j_{n-1}} & 2\widetilde{B}_{11} & \cdots & 2\widetilde{B}_{nn} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(n-1)\widetilde{B}^{n-2}_{i_1 j_1} & \cdots & (n-1)\widetilde{B}^{n-2}_{i_{n-1} j_{n-1}} & (n-1)\widetilde{B}^{n-2}_{11} & \cdots & (n-1)\widetilde{B}^{n-2}_{nn}
\end{array}\right]
$$

Jacobian

Lemma

Let A have the Duarte property with respect to the vertex 1, $G(A)$ be a tree T, and X be a symmetric matrix such that

1. $I \circ X = O$,
2. $A \circ X = O$,
3. $[A, X](1) = O$.

then $X = O$.

Jacobian

Assume $c^T \text{jac}(F)\Big|_A = 0$ for some c

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Let $Y := p(A) + \widetilde{q(B)}$ then $Y \circ A = 0$ and $Y \circ I = O$ and

$$[A, p(A)] + [A, \widetilde{q(B)}] = [A, Y]$$

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Since $[A, p(A)] = O$,

$$[A, Y] = [A, \widetilde{q(B)}] = \left[\begin{array}{c|ccc} * & * & \cdots & * \\ \hline * & & & \\ \vdots & & O & \\ * & & & \end{array}\right]$$

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By Lemma $Y = O$

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$$\tilde{B}X_{\cdot,j} = \left[\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right] \left[\begin{array}{c} 0 \\ \hline x_2 \\ \vdots \\ x_n \end{array}\right] = 0 \Rightarrow B \begin{bmatrix} x_2 \\ \vdots \\ x_n \end{bmatrix} = 0$$

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So $X = O$, hence $c = O$. That is, $\mathrm{Jac}(F)\Big|_A$ is invertible.

The Implicit Function Theorem

Theorem

$x \in \mathbb{R}^s,\ y \in \mathbb{R}^r$

$F : \mathbb{R}^{s+r} \to \mathbb{R}^s$: *continuously differentiable on an open subset* U *of* $\mathbb{R}^{s+r}$

$$F(x, y) = (F_1(x, y), F_2(x, y), \ldots, F_s(x, y)),$$

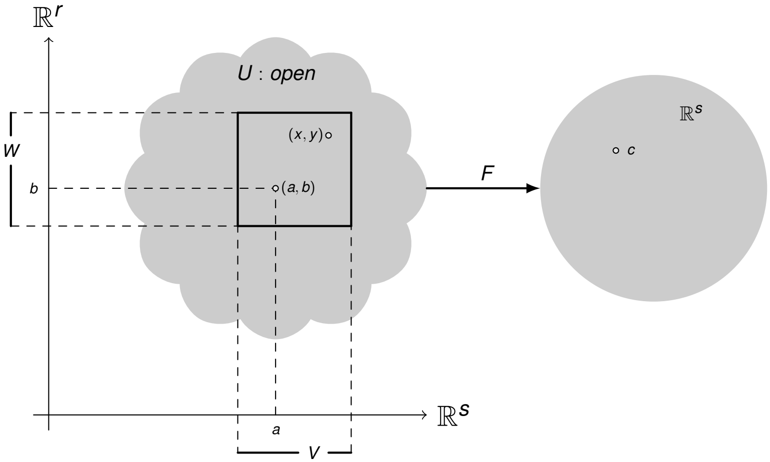
$(a, b) \in U$ *with* $a \in \mathbb{R}^s,\ b \in \mathbb{R}^r$

$c \in \mathbb{R}^s$ *such that* $F(a, b) = c$

If $\left[\frac{\partial F_i}{\partial x_j} \Big|_{(a,b)} \right]$ *is nonsingular, then there exist an open neighborhood* V *containing* a *and an open neighborhood* W *containing* b *such that* $V \times W \subseteq U$ *and for each* $y \in W$ *there is an* $x \in V$ *with*

$$F(x, y) = c$$

The Implicit Function Theorem



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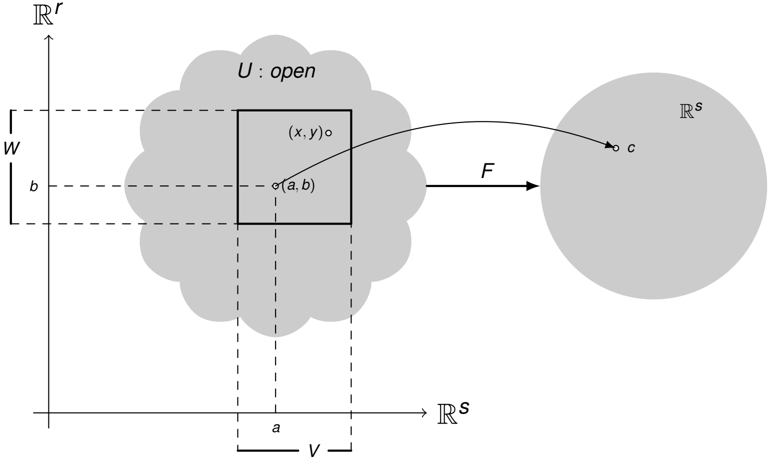

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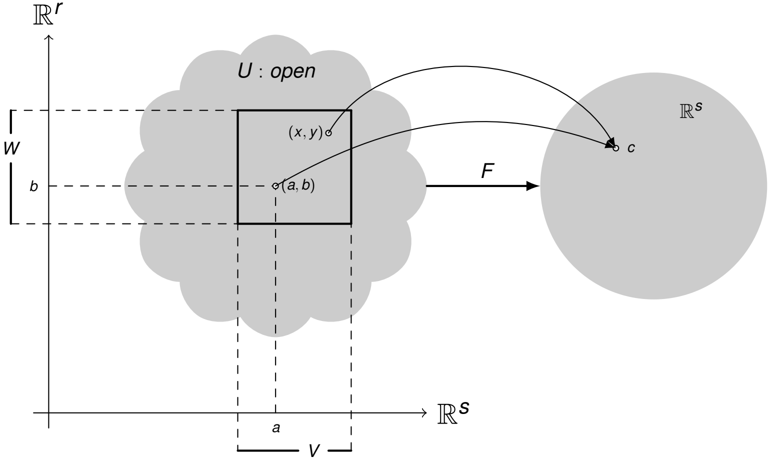

Future Research

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- Known: Any connected graph on n vertices realizes

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and

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Is there a matrix A with $G(A) = G$ and eigenvalues λ_i's such that $A(1)$ has eigenvalues μ_i's and $A(\{1,2\})$ has eigenvalues γ_i's?

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- What about trees and graphs instead of Jacobi matrices?

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  - T has eigenvalues $\lambda_1, \ldots, \lambda_m$
  - The leading $(j-1) \times (j-1)$ principal submatrix has eigenvalues $\alpha_1 \leq \cdots \leq \alpha_{j-1}$
  - The trailing $(n-j) \times (n-j)$ principal submatrix has eigenvalues $\beta_1 \leq \cdots \leq \beta_{n-j}$

  if and only if
  - $\{\alpha_i\} \cap \{\beta_i\} = \emptyset$
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Thank You!!