The Jacobian Method

The Art of Finding More Needles in Nearby Haystacks

A Ph.D. defense by

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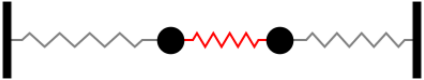
Advisor: Bryan L. Shader

University of Wyoming, July 2014



Animation courtesy of Dr. Dan Russell, Grad. Prog. Acoustics, Penn State

Introduction:

History, motivation
Definitions and preliminaries

Often for mathematicians finding a needle in a haystack can be formulated as solving

$$f(x, y) = c,$$

where $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$

The Implicit Function Theorem says when a particular solution $f(x_0, y_0) = c$ is ‘nice’ then one can solve $f(x, y_1) = c$ for any y_1 near y_0.

i.e. if you find a nice needle in the haystack $f(x, y_0) = c$,

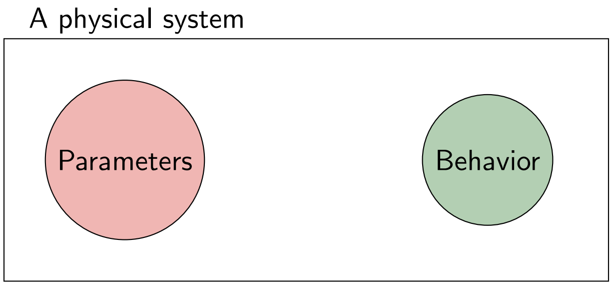
then all nearby haystacks $f(x, y_1) = c$ have a needle.

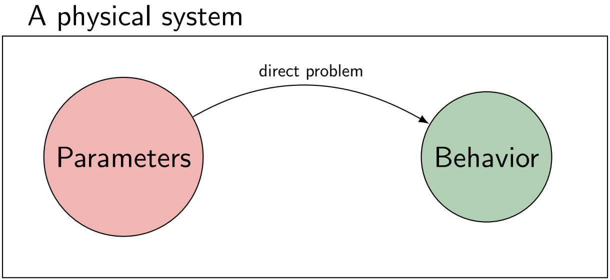
Why do we care?

B. Parlett,
in *The Zahir*

“Vibrations are everywhere, and so too are the eigenvalues associated with them.”

Beresford N. Parlett, 1998


A physical system
Parameters
Behavior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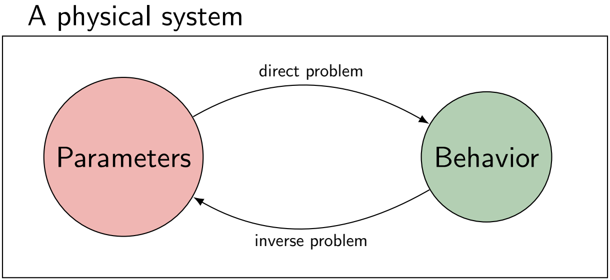
A physical system
direct problem
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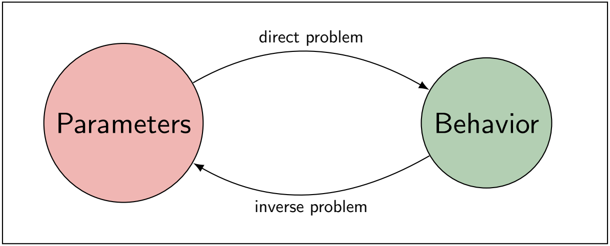
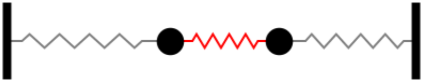
A physical system



- Agustin Cauchy (1789–1875)
- Jacques Sturm (1803–1855)
- Joseph Liouville (1809–1882)

A physical system:



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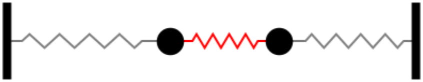

Motion equations:

$$m_1\ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) + F_1(t),$$

$$m_2\ddot{x}_2 = -k_3 x_2 - k_2(x_2 - x_1) + F_2(t).$$

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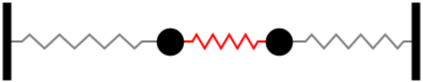

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Matrix form:

$$\begin{bmatrix} m_1 & \\ & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}.$$

$$A\ddot{\mathbf{x}} + C\mathbf{x} = \mathbf{F}$$

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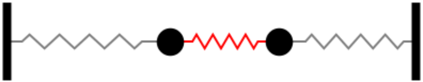

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$$A\ddot{\mathbf{x}} + C\mathbf{x} = \mathbf{F}$$

When $\mathbf{F} = \mathbf{0}$, the eigenvalues of C describe the 'natural frequencies' of the system.

Graph of a matrix

$A_{n \times n}$: real symmetric matrix
$G(A)$: a graph G on n vertices $1, 2, \ldots, n$
$i \sim j$ if and only if $i \neq j$ and $a_{ij} \neq 0$

$G(A)$ does not depend on the diagonal entries of A

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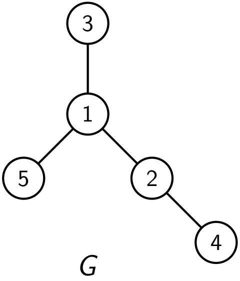

$$A = \begin{bmatrix} 2 & 1 & 3 & 0 & -4 \\ 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Then we say $A \in S(G)$.

- In general the coefficient matrix C for the system of masses and springs has a certain zero-nonzero pattern that can be described by its graph.

$$C = \begin{bmatrix} k_1 + k_2 & -k_2 & & \\ -k_2 & k_2 + k_3 & -k_3 & \\ & -k_3 & k_3 + k_4 & -k_4 \\ & & -k_4 & k_4 + k_5 \end{bmatrix}$$

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Previous studies when graph of A is a:

- star [Fan, Pall 1957]

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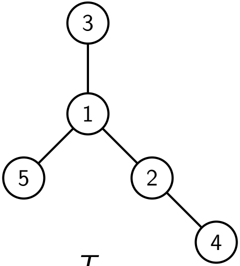
Previous studies when graph of A is a:

- star [Fan, Pall 1957]
- path [Gladwell 1988]
- tree [Duarte 1989]

Submatrices and Subtrees



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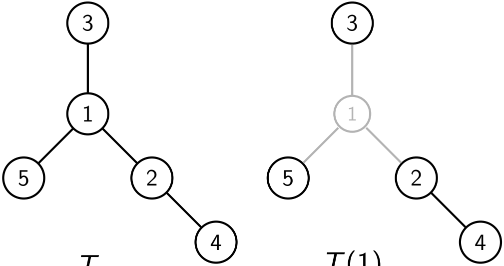
Submatrices and Subtrees



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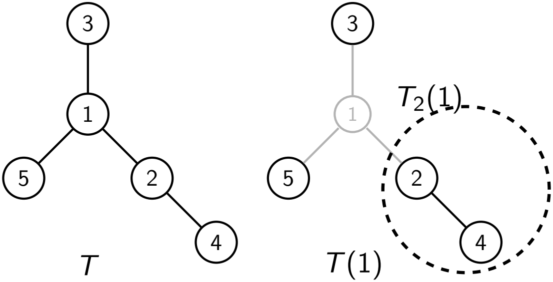
Submatrices and Subtrees



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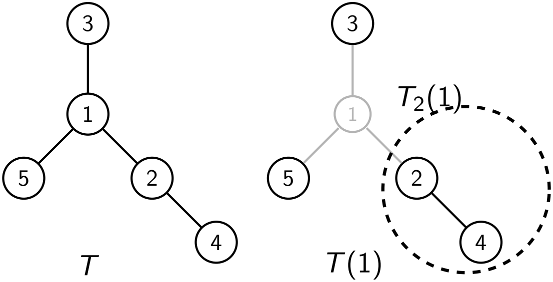

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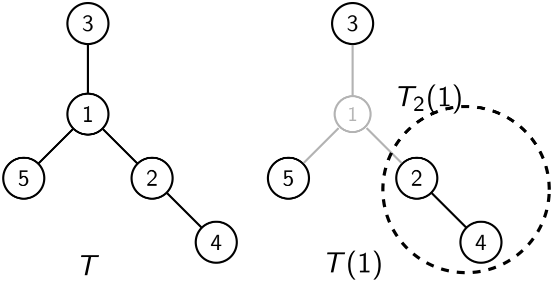

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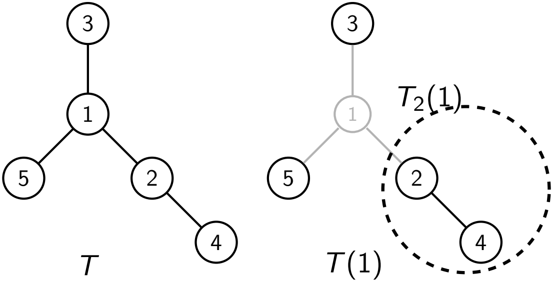

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Cauchy Interlacing Inequalities

$A_{n \times n}$: real symmetric matrix
Eigenvalues of A: $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$
Eigenvalues of $A(r)$: $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$
Eigenvalues of $A(\{r, s\})$: $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_{n-2}$

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$$\lambda_i \leq \mu_i \leq \lambda_{i+1}, \qquad i = 1, \ldots, n-1,$$

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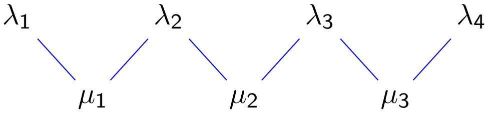

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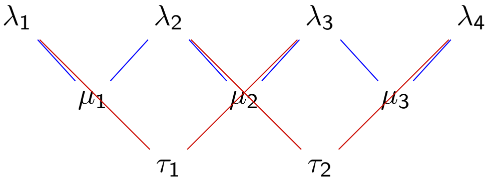


The Implicit Function Theorem

Theorem

$\mathbf{x} \in \mathbb{R}^s$, $\mathbf{y} \in \mathbb{R}^r$

$F : \mathbb{R}^{s+r} \to \mathbb{R}^s$: *continuously differentiable on an open subset* U *of* $\mathbb{R}^{s+r}$

$$F(\mathbf{x}, \mathbf{y}) = (F_1(\mathbf{x}, \mathbf{y}), F_2(\mathbf{x}, \mathbf{y}), \ldots, F_s(\mathbf{x}, \mathbf{y})),$$

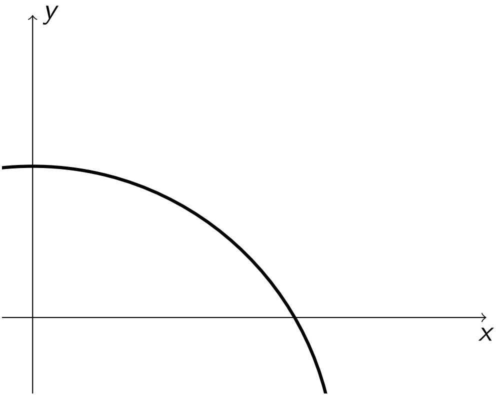
$(\mathbf{a}, \mathbf{b}) \in U$ *with* $\mathbf{a} \in \mathbb{R}^s$, $\mathbf{b} \in \mathbb{R}^r$

$\mathbf{c} \in \mathbb{R}^s$ *such that* $F(\mathbf{a}, \mathbf{b}) = \mathbf{c}$

If $\left[\frac{\partial F_i}{\partial x_j} \bigg|_{(a,b)} \right]$ *is nonsingular, then there exist an open neighborhood* V *containing* $\mathbf{a}$ *and an open neighborhood* W *containing* $\mathbf{b}$ *such that* $V \times W \subseteq U$ *and for each* $\mathbf{y} \in W$ *there is an* $\mathbf{x} \in V$ *with*

$$F(\mathbf{x}, \mathbf{y}) = \mathbf{c}$$

The Implicit Function Theorem



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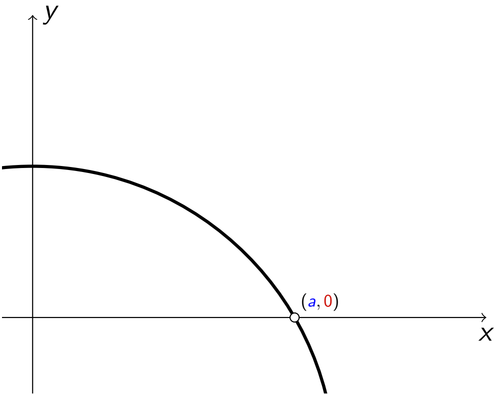

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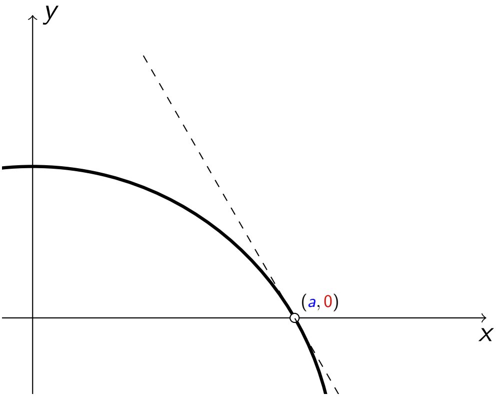

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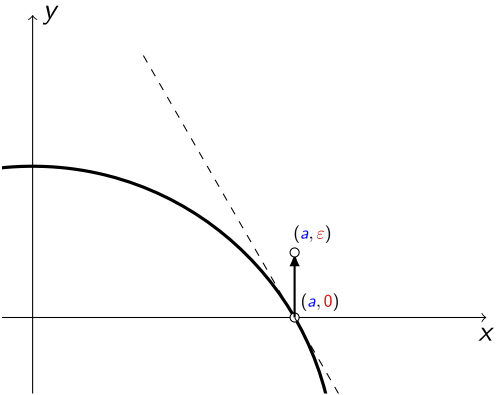

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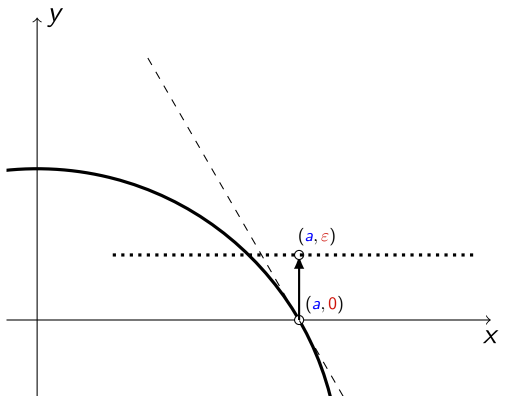

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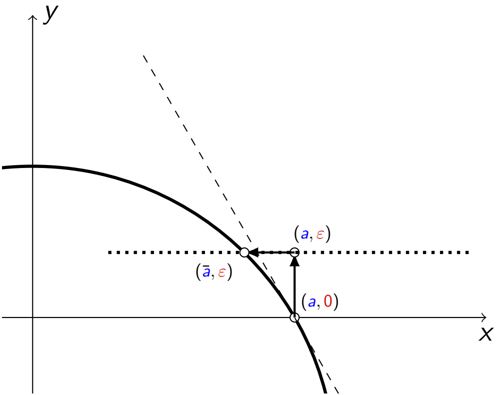

The
Jacobian Method

The λ-Structured Inverse Eigenvalue Problem

Theorem (K. H.M. and B.L. Shader 2014)

For given

$\lambda_1 < \lambda_2 < \cdots < \lambda_n$*: real numbers, and*

G*: a graph on* n *vertices,*

there is an $n \times n$ *real symmetric matrix* A *such that*

eigenvalues of A *are* $\lambda_1, \lambda_2, \ldots, \lambda_n$*, and*

graph of A *is* G*.*

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Proof:

Let

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Define:

$$M = M(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} x_1 & y_1 & \cdots & \\ y_1 & x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & y_m \\ & \cdots & y_m & x_n \end{bmatrix}$$

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Note that $M(\lambda_1, \ldots, \lambda_n, 0, \ldots, 0) = A$.

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Define:

$$F : (\mathbf{x}, \mathbf{y}) \mapsto \big(\lambda_1(M), \lambda_2(M), \cdots, \lambda_n(M)\big).$$

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Define:

$$F : (\mathbf{x}, \mathbf{y}) \mapsto \big(\lambda_1(M), \lambda_2(M), \cdots, \lambda_n(M)\big).$$

Note that $F\Big|_A = F(\lambda_1, \ldots, \lambda_n, 0, \ldots, 0) = (\lambda_1, \lambda_2, \cdots, \lambda_n)$.

$$F : (\mathbf{x}, \mathbf{y}) \mapsto \big(\lambda_1(M), \lambda_2(M), \cdots, \lambda_n(M)\big).$$

Then

$$\mathrm{Jac}(F)\Big|_A = \left[\begin{array}{c|c} I & O \end{array}\right]$$

has full row-rank.

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$$F(\lambda_1, \ldots, \lambda_n, 0, \ldots, 0) = (\lambda_1, \lambda_2, \cdots, \lambda_n).$$

Then by the Implicit Function Theorem, for $\bar{\mathbf{y}} = (\varepsilon_1, \ldots, \varepsilon_m)$ with sufficiently small $\varepsilon_i > 0$, there is an $\bar{\mathbf{x}} = (\overline{\lambda_1}, \ldots, \overline{\lambda_n})$ close to $(\lambda_1, \ldots, \lambda_n)$ such that

$$F(\overline{\lambda_1}, \ldots, \overline{\lambda_n}, \varepsilon_1, \ldots, \varepsilon_m) = (\lambda_1, \lambda_2, \cdots, \lambda_n).$$

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Let $\bar{A} = M(\bar{\mathbf{x}}, \bar{\mathbf{y}})$. Then graph of $\bar{A}$ is G, and eigenvalues of $\bar{A}$ are $\lambda_1, \ldots, \lambda_n$. $\square$

Three Fundamental Structured Inverse Eigenvalue Problems

The λ-μ-SIEP: for trees

Theorem (A. Leal-Duarte 1989)

For given

$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \mu_{n-1} < \lambda_n$*: real numbers,*

G*: a tree on* n *vertices, and*

v*: a fixed vertex of* G*,*

there is an $n \times n$ *real symmetric matrix* A *such that*

eigenvalues of A *are* $\lambda_1, \lambda_2, \ldots, \lambda_n$*,*

eigenvalues of $A(v)$ *are* $\mu_1, \mu_2, \ldots, \mu_{n-1}$*, and*

graph of A *is* G*.*

The λ-μ-SIEP: for connected graphs

Theorem (K. H.M. and B.L. Shader 2013)

For given

$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \mu_{n-1} < \lambda_n$*: real numbers,*

G*: a connected graph on* n *vertices, and*

v*: a fixed vertex of* G*,*

there is an $n \times n$ *real symmetric matrix* A *such that*

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eigenvalues of $A(v)$ *are* $\mu_1, \mu_2, \ldots, \mu_{n-1}$*, and*

graph of A *is* G*.*

The λ-μ-SIEP: perturbing a diagonal entry

Theorem (K. H.M. and B.L. Shader 2014)

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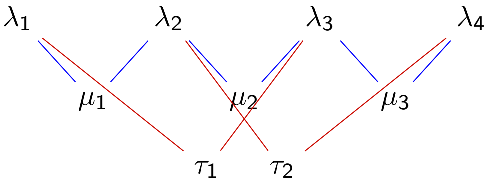
eigenvalues of $A + aE_{vv}$ *are* $\mu_1, \mu_2, \ldots, \mu_n$*, and*

graph of A *is* G*,*

where $a = \sum_i (\mu_i - \lambda_i)$*.*

A τ-pairing

Recall the second order Cauchy interlacing inequalities:



If two consecutive τ's are between two consecutive λ's, it is called a τ-pairing

A τ-pairing

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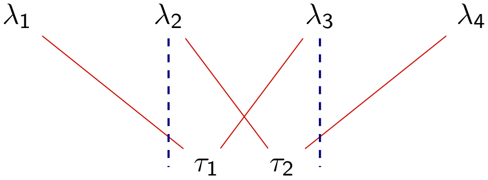

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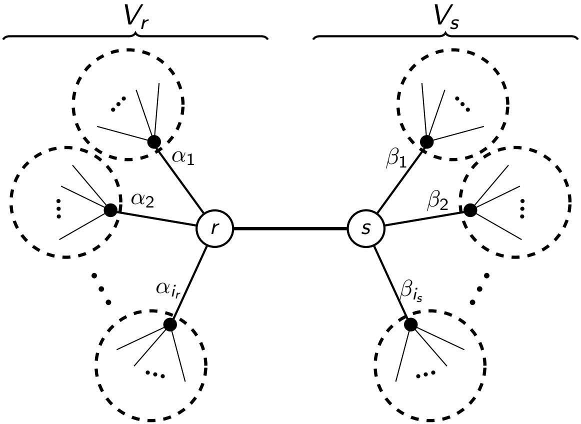
A tree with adjacent vertices r and s



The λ-τ-SIEP: for trees

Theorem (K. H.M. and B.L. Shader 2014)

For given

$\lambda_1 < \lambda_2 < \cdots < \lambda_n$ *and* $\tau_1 < \tau_2 < \cdots < \tau_{n-2}$*: real numbers,*

G*: a tree on n vertices, and*

r, s*: two vertices of* G*,*

where

$\lambda_i < \tau_i < \lambda_{i+2}$*, and* $\tau_i \neq \lambda_{i+1}$ *for* $i = 1, \ldots, n-2$*,*

there are k τ*-pairings, and*

$T\left[V_r \setminus \{r\}\right]$ *and* $T\left[V_s \setminus \{s\}\right]$ *each have at least* k *vertices,*

there is an $n \times n$ *real symmetric matrix* A *such that*

eigenvalues of A *are* $\lambda_1, \lambda_2, \ldots, \lambda_n$*,*

eigenvalues of $A(\{r, s\})$ *are* $\tau_1, \tau_2, \ldots, \tau_{n-2}$*, and*

graph of A *is* G*.*

The λ-τ-SIEP: for connected graphs

Theorem (K. H.M. and B.L. Shader 2014)

Under the same assumptions, if there are k τ-pairings, and G has a spanning tree T as before, then there is an $n \times n$ real symmetric matrix A such that

eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_n$,

eigenvalues of $A(\{r, s\})$ are $\tau_1, \tau_2, \ldots, \tau_{n-2}$, and

graph of A is G.

The λ-τ-SIEP: perturbing two diagonal entries

Theorem (K. H.M. and B.L. Shader 2014)

Under the same assumptions, if there are k τ-pairings, and G has a spanning tree T as before, for given

$\lambda_1 < \lambda_2 < \cdots < \lambda_n$ *and* $\tau_1 < \tau_2 < \cdots < \tau_n$*: real numbers,*

where

$\lambda_i < \tau_i < \lambda_{i+2}$, *and* $\tau_i \neq \lambda_{i+1}$ *for* $i = 1, \ldots, n-2$,

$\lambda_j < \tau_j$ *for* $j = n-1, n-2$,

there is an $n \times n$ *real symmetric matrix A and real numbers* a_r *and* a_s *such that*

eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_n$,

eigenvalues of $A + a_r E_{rr} + a_s E_{ss}$ *are* $\tau_1, \tau_2, \ldots, \tau_n$, *and*

graph of A is G.

The Nowhere-zero Eigenbasis SIEP: for trees

Theorem (K. H.M. and B.L. Shader 2014)

For given

$\lambda_1 < \lambda_2 < \cdots < \lambda_n$*: real numbers, and*

G*: a tree on* n *vertices,*

there is an $n \times n$ *real symmetric matrix* A *such that*

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graph of A *is* G*, and*

none of the eigenvectors of A *has a zero entry.*

The Nowhere-zero Eigenbasis SIEP: for connected graphs

Theorem (K. H.M. and B.L. Shader 2014)

For given

$\lambda_1 < \lambda_2 < \cdots < \lambda_n$*: real numbers, and*

G*: a connected graph on* n *vertices,*

there is an $n \times n$ *real symmetric matrix* A *such that*

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graph of A *is* G*, and*

none of the eigenvectors of A *has a zero entry.*

Overcoming Difficulties

Derivatives of F

Consider the maps

$$F : M \mapsto (\lambda_1(M), \lambda_2(M), \ldots, \lambda_n(M)),$$

where $\lambda_i(M)$ is the i-th smallest eigenvalue of M.

$$G : M \mapsto (c_0(M), c_1(M), \ldots, c_{n-1}(M)),$$

where $c_i(M)$ is the coefficient of x^i in the characteristic polynomial of M.

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$$f : M \mapsto (\operatorname{tr} M, \operatorname{tr} M^2, \ldots, \operatorname{tr} M^n).$$

Then,

$$\frac{\partial}{\partial x_t}(\operatorname{tr} M^k) = 2k \left(M^{k-1}\right)_{i,j}.$$

Then

$$\mathrm{Jac}(f)\Big|_A = \left[\begin{array}{ccc|ccc} I_{11} & \cdots & I_{nn} & I_{i_1 j_1} & \cdots & I_{i_m j_m} \\ A_{11} & \cdots & A_{nn} & A_{i_1 j_1} & \cdots & A_{i_m j_m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{11}^{n-1} & \cdots & A_{nn}^{n-1} & A_{i_1 j_1}^{n-1} & \cdots & A_{i_m j_m}^{n-1} \end{array}\right],$$

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and

$$\mathrm{Jac}_x(f)\Big|_A = \left[\begin{array}{ccc|ccc} I_{11} & \cdots & I_{nn} & I_{i_1 j_1} & \cdots & I_{i_m j_m} \\ A_{11} & \cdots & A_{nn} & A_{i_1 j_1} & \cdots & A_{i_m j_m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{11}^{n-1} & \cdots & A_{nn}^{n-1} & A_{i_1 j_1}^{n-1} & \cdots & A_{i_m j_m}^{n-1} \end{array}\right].$$

Nonsingularity of $\mathrm{Jac}_{\mathbf{x}}(\mathbf{f})\big|_{\mathbf{A}}$

In the λ-μ-SIEP for connected graphs let A be a solution for the λ-μ-SIEP for a spanning tree T of G, $B = A(v)$, and let

$$M = \begin{bmatrix} 2x_1 & x_{n+1} & y_1 & \cdots & \\ x_{n+1} & 2x_2 & x_{n+2} & \ddots & \vdots \\ y_1 & x_{n+2} & 2x_3 & \ddots & y_m \\ \vdots & \ddots & \ddots & \ddots & x_{2n-1} \\ & \cdots & y_m & x_{2n-1} & 2x_n \end{bmatrix}, N = M(v).$$

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Define

$$f(\mathbf{x}, \mathbf{y}) := \left(\frac{\operatorname{tr} M}{2}, \frac{\operatorname{tr} M^2}{4}, \ldots, \frac{\operatorname{tr} M^n}{2n}, \frac{\operatorname{tr} N}{2}, \frac{\operatorname{tr} N^2}{4}, \ldots, \frac{\operatorname{tr} N^{n-1}}{2(n-1)} \right).$$

Let $\widetilde{B}$ be the matrix obtained from B by inserting a zero row and column in the v-th row and column of it. Then

$$\mathrm{Jac}_x(f)\Big|_A = \left[\begin{array}{ccc|ccc} I_{11} & \cdots & I_{nn} & I_{i_1j_1} & \cdots & I_{i_{n-1}j_{n-1}} \\ A_{11} & \cdots & A_{nn} & A_{i_1j_1} & \cdots & A_{i_{n-1}j_{n-1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{11}^{n-1} & \cdots & A_{nn}^{n-1} & A_{i_1j_1}^{n-1} & \cdots & A_{i_{n-1}j_{n-1}}^{n-1} \\ \hline \widetilde{I}_{11} & \cdots & \widetilde{I}_{nn} & \widetilde{I}_{i_1j_1} & \cdots & \widetilde{I}_{i_{n-1}j_{n-1}} \\ \widetilde{B}_{11} & \cdots & \widetilde{B}_{nn} & \widetilde{B}_{i_1j_1} & \cdots & \widetilde{B}_{i_{n-1}j_{n-1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{B}_{11}^{n-2} & \cdots & \widetilde{B}_{nn}^{n-2} & \widetilde{B}_{i_1j_1}^{n-2} & \cdots & \widetilde{B}_{i_{n-1}j_{n-1}}^{n-2} \end{array}\right].$$

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$$\det(\mathrm{Jac}_x(f)\Big|_A) = \quad ?$$

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$$\det(\mathrm{Jac}_x(f)\Big|_A) = \quad ?$$

Solution: Rows of $\mathrm{Jac}_x(f)\Big|_A$ are linearly independent.

Assume $\boldsymbol{\alpha}^T \operatorname{Jac}_x(f)\Big|_A = \mathbf{0}$. i.e.

$$\alpha_1 \operatorname{Jac}_1 + \cdots + \alpha_{2n-1} \operatorname{Jac}_{2n-1} = \mathbf{0} \qquad (*)$$

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Define:

$$p(x) = \alpha_1 + \alpha_2 x + \cdots + \alpha_n x^{n-1},$$

$$q(x) = \alpha_{n+1} + \alpha_{n+2} x + \cdots + \alpha_{2n-1} x^{n-2}.$$

and

$$X = p(A) + \widetilde{q(B)}.$$

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Then

$$(*) \iff \begin{cases} X \circ A = O, \\ X \circ I = O. \end{cases}$$

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Also note that $(AX - XA)(v) = O$.

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and

$$X = p(A) + \widetilde{q(B)}.$$

Then

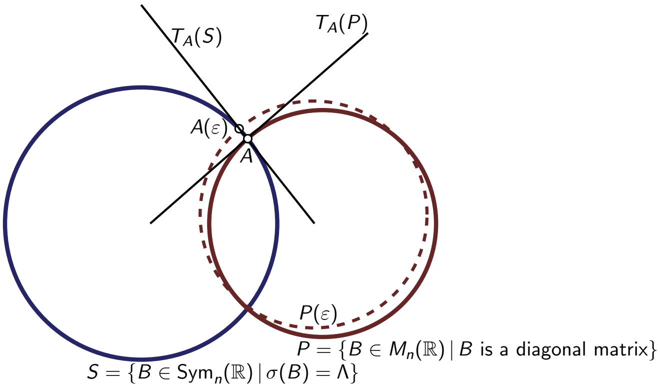
$$(*) \iff \begin{cases} X \circ A = O, \\ X \circ I = O. \end{cases}$$

Also note that $(AX - XA)(v) = O$. It can be shown that $X = O$, and $p(x) \equiv 0$, $q(x) \equiv 0$.
Hence $\boldsymbol{\alpha} = \mathbf{0}$.

Future Work

Transverse Intersections

The mentioned problem can be described in terms of some manifolds intersecting transversally, which gives a more general approach to the Jacobian method.



Question: Could this help us to solve the cases where there are repeated eigenvalues, or the interlacing inequalities are not strict?

The Constant Rank Theorem vs. IFT

Theorem (Constant Rank Theorem[1])

Assume $a \in U \subseteq \mathbb{R}^n$, $F = (f_1, \dots, f_m) : U \to \mathbb{R}^m$ is $C^\infty(U)$, and the rank of $\mathrm{Jac}(F)\Big|_x$ is k for all x in a neighborhood of a. Then there are open neighborhoods V of a and W of $F(a)$ and diffeomorphisms $\phi : V \to \mathbb{R}^n$ and $\psi : W \to \mathbb{R}^m$ with

$$\begin{array}{ccc} V & \xrightarrow{F} & W \\ \phi \downarrow & & \downarrow \psi \\ \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \end{array}$$

such that $\psi \circ F \circ \phi^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_k, 0, \dots, 0)$.

Question: Could this help us to solve the cases where there are repeated eigenvalues, or the interlacing inequalities are not strict?

[1] [S.G. Krantz and H.R. Parks, The Implicit Function Theorem: History, Theory, and Applications, 2013]

λ-SIEP for G when multiplicities are allowed

Question: What if the λ_i's are not distinct?

$$A = \left[\overbrace{\begin{matrix} \times & \times & & & & \\ \times & \times & \times & & & \\ & \times & \times & \times & & \\ & & \times & \times & \times & \\ & & & \times & \times & \times \\ & & & & \times & \times \end{matrix}}^{\Lambda} \right]$$

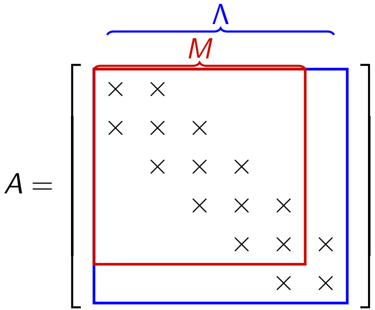
λ-μ-SIEP for G when multiplicities are allowed

Questions: What if some of these conditions are **not** necessary?

The λ_i's are distinct.

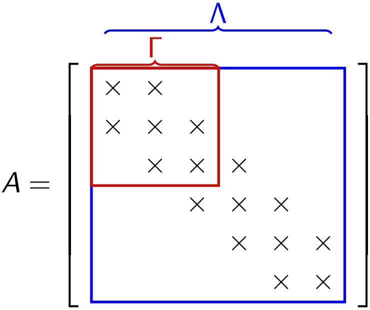
The μ_i's are distinct.

The μ_i's **strictly** interlace the λ_i's.



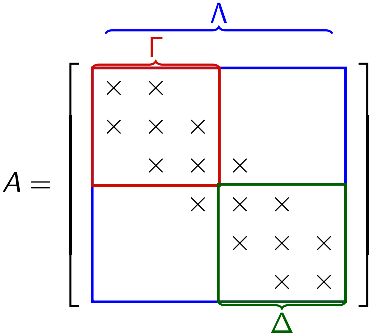
Generalization: λ-γ-SIEP for G when multiplicities are allowed

Question: What about the case that the eigenvalues of G and a $k \times k$ principal submatrix of it are prescribed?



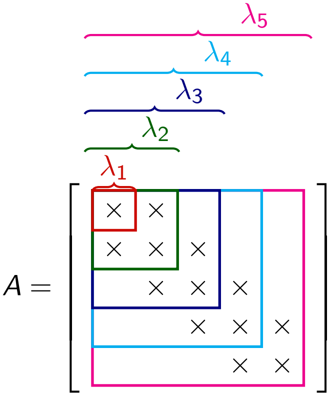
Generalization: λ-γ-δ-SIEP for G when multiplicities are allowed

Question: What about the case that the eigenvalues of G and a $k \times k$ principal submatrix of it and its complement are prescribed?



Other Problems

Question: Let G be a graph on n vertices and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be n real numbers. Is there a real symmetric matrix A such that $G(A) = G$ and $\lambda_k \in \sigma(A[1, 2, \ldots, k])$, for $k = 1, 2, \ldots, n$?

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