On the permanent rank of matrices

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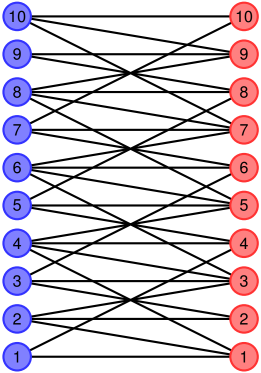
Outline

1. **Permanent**
   - Examples
   - Definition
2. **Permanent rank**
   - Examples
   - Definition
   - Yu's conjecture
   - AJT conjecture
   - perrank and termrank

How many perfect matchings are in a bipartite graph?



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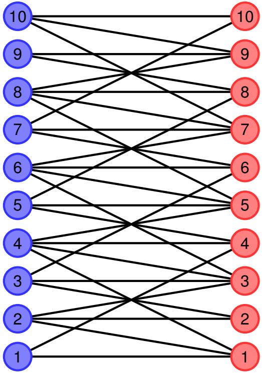

$$\left[\begin{array}{cc|cc|cc|cc|cc}
1 & 0 & 0 & 1 & & & & & & \\
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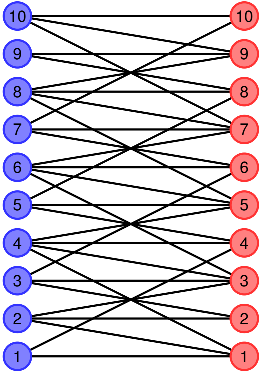
How many perfect matchings are in a bipartite graph?



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$$\text{per}(A) = 95$$

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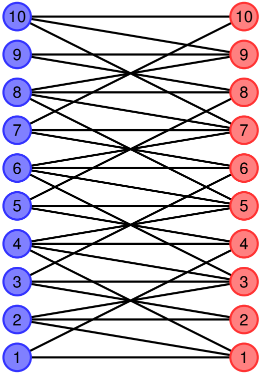
- No easy* geometric interpretation
- Appears naturally in many combinatorial settings

∗ such as the volume of a parallelepiped

What is the greatest matching in a bipartite graph?



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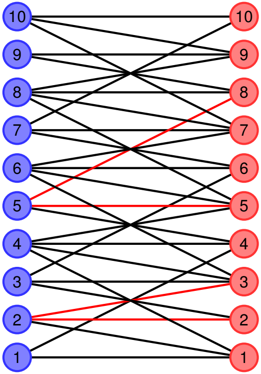

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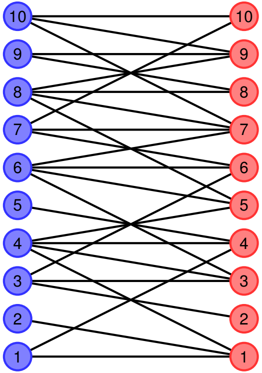

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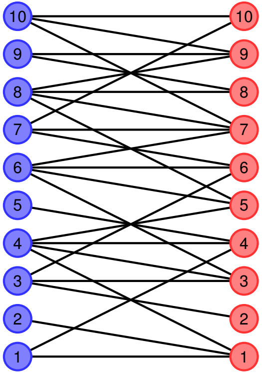

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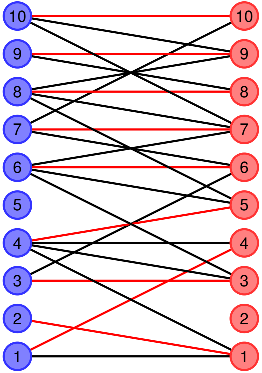

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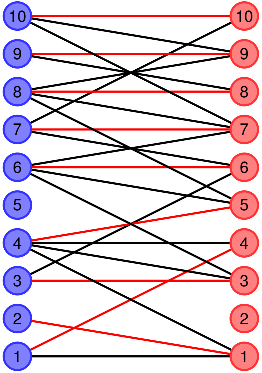

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$$\text{per}(A') = 18$$

Definition

The perrank *of a matrix A is the size of a largest sub-matrix of A with nonzero permanent*

Yu's results

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- For full matrices A, B over a characteristic zero field $\mathbb{F}$:

$$\operatorname{perrank}\left(\begin{array}{c|c} A & B \end{array}\right) = n$$

Yu's conjecture (The Permanent conjecture)

Conjecture

For an invertible matrix A:

$$\text{perrank}\left(\, A \, \middle| \, A \, \right) = n$$

AJT conjecture

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$\mathbb{F}$: *a field with at least 4 elements*

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$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\prod x_i \prod y_i \neq 0$$

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$$\Rightarrow \prod x_i \prod A_i x \neq 0$$

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$$x_i \in \{1, 2\}$$

Term rank of a matrix

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$$\text{termrank}(A) = 4$$

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perrank-termrank

Conjecture (B.L.S.)

$$\text{perrank}(A) \geq \left\lceil \frac{\text{termrank}(A)}{2} \right\rceil$$

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For even termrank *the equality holds iff*

$$A \simeq \bigoplus \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Questions

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- AJT conjecture is true over $\mathbb{F}_5$
- Converting computing the permanent of an $n \times n$ matrix to computing the determinant of a matrix of polynomial size in n.

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