An Analogue of Matrix Tree Theorem for Signless Laplacians

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Definitions & Notations

For a given graph G on n vertices $1, 2, \ldots, n$ let

- A: Adjacency matrix
- D: Diagonal matrix of the degrees
- $L = D - A$: Laplacian matrix
- $Q = D + A$: Signless Laplacian matrix

of G.

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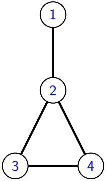
Example



$$\overset{A}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}} \quad \overset{D}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}} \quad \overset{L}{\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}} \quad \overset{Q}{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}}$$

What is known

Theorem

- *G: simple graph on n vertices*
- *Spectrum of L: $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$*
- *Spectrum of Q: $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$*

Then G is bipartite if and only if

$$\{\mu_1, \mu_2, \ldots, \mu_n\} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}.$$

What is known

Theorem

- *G: simple graph on n vertices*

Then

- *Multiplicity of 0 as an eigenvalue of L is equal to the number of connected components of G.*
- *Multiplicity of 0 as an eigenvalue of Q is equal to the number of bipartite connected components of G.*

What is known

Theorem (Matrix-Tree Theorem)

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- *Spectrum of L: $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$*
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Then the number of spanning trees of G, $t(G)$ is

$$t(G) = \det(L(i)) = \frac{\mu_2 \cdots \mu_n}{n},$$

for all $i = 1, 2, \ldots, n$.

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Theorem (Matrix-Tree Theorem)

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Question

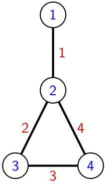
$$\det(Q(i)) = ?$$

An observation

In general

$$\det(Q(i)) \neq \det(Q(j))$$

Example



$$Q$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

$$\det(Q(1)) = 7$$

$$\det(Q(2)) = 3$$

$$\det(Q(3)) = 3$$

$$\det(Q(4)) = 3$$

Definitions & Notations

For a given graph G on n vertices $1, 2, \ldots, n$ and m vertices $1, 2, \ldots, m$ let

- $N_{n \times m}$: Incidence matrix
- $N'_{n \times m}$ Incidence matrix of an orientation of the edges

of G.

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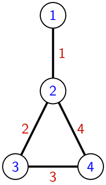

N

	1	2	3	4
1	1	0	0	0
2	1	1	0	1
3	0	1	1	0
4	0	0	1	1

N'

	1	2	3	4
1	1	0	0	0
2	−1	1	0	1
3	0	−1	1	0
4	0	0	−1	−1

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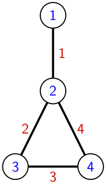

$$N = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

$$N' = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{array}\right] \end{array}$$

Then

- $L = N' N'^{\top}$,
- $Q = NN^{\top}$.

Binet-Cauchy

Theorem

Let $m \leq n$. For $m \times n$ matrices A and B, we have

$$\det(AB^\top) = \sum_S \det(A(;S])\det(B(;S]),$$

where the summation runs over all m-subsets S of $\{1, 2, \ldots, n\}$.

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$$\begin{aligned}\det(Q) = \det(NN^\top) &= \sum_S \det(N(;S]) \det(N(;S]) \\ &= \sum_S \det(N(;S])^2.\end{aligned}$$

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And

$$\det(Q(i)) = \sum_S \det(N(i;S])^2.$$

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Lemma

If H is a TU-subgraph on n vertices with $n - k$ edges consisting of c odd-unicyclic graphs and s trees, then $s = k$.

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If G is an odd (resp. even) unicyclic graph, then the determinant of its incidence matrix is ± 2 (resp. zero).

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Let H be a tree with at least one edge and N be the incidence matrix of H. Then $\det(N(i;)) = \pm 1$ for all vertices i of H.

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Lemma

Let H be a graph on n vertices and $n-1$ edges with incidence matrix N. If H has a connected component which is a tree and an edge which is not on the tree, then $\det(N(i;)) = 0$ for all vertices i not on the tree.

TU-subgraphs

Lemma

Let H be a spanning subgraph of a graph on n vertices with edges indexed by S and $|S| = n - 1$. Then one of the following is true.

1. *H is a tree.*
2. *H has an even cycle and a vertex not on the cycle.*
3. *H has no even cycles, but H has a connected component with at least two odd cycles and at least two connected components which are trees.*
4. *H is a disjoint union of c odd unicyclic graphs and exactly one tree, i.e., H is a TU-graph.*

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Let H be a spanning subgraph of a graph on n vertices with edges indexed by S and $|S| = n - 1$. Then one of the following is true.

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   $$\Rightarrow \begin{cases} \det(N(i;S]) = \quad 0 & ; i \text{ is not on the tree} \\ \det(N(i;S]) = \pm 2^c & ; i \text{ is on the tree} \end{cases}$$

Main result

Theorem

- *G: a simple connected graph on n vertices $1, 2, \ldots, n$*
- *Q: the signless Laplacian matrix of G*

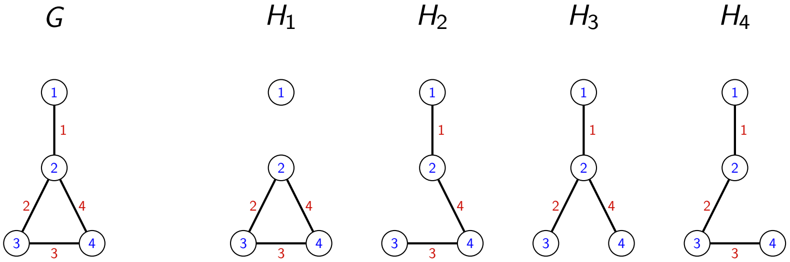
Then

$$\det(Q(i)) = \sum_{H} 4^{c(H)},$$

where the summation runs over all TU-subgraphs H of G with $n-1$ edges consisting of a unique tree on vertex i and $c(H)$ odd-unicyclic graphs.

Main result

Example



$c(H_1) = 1 \quad c(H_2) = 0 \quad c(H_3) = 0 \quad c(H_4) = 0$

$$\det(Q(1)) = \sum_{i=1}^{4} 4^{c(H_i)} = 4^1 + 4^0 + 4^0 + 4^0 = 7.$$

Main result

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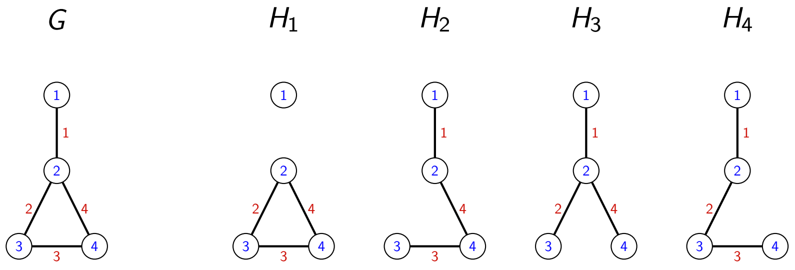

$$c(H_1) = 1 \quad c(H_2) = 0 \quad c(H_3) = 0 \quad c(H_4) = 0$$

$$\det(Q(1)) = \sum_{i=1}^{4} 4^{c(H_i)} = 4^1 + 4^0 + 4^0 + 4^0 = 7.$$

$$\det(Q(2)) = \qquad 4^0 + 4^0 + 4^0 = 3.$$

Main results

Corollary

- *G: a simple connected graph on n vertices $1, 2, \ldots, n$*
- *Q the signless Laplacian matrix of G*
- *$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$: eigenvalues of Q*

Then

(a) $t(G) \leq \det(Q(i))$
the equality holds if and only if all odd cycles of G contain vertex i.

(b) $$t(G) \leq \frac{1}{n} \sum_{1 \leq i_1 < i_2 < \cdots < i_{n-1} \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_{n-1}}$$
the equality holds if and only if G is an odd cycle or a bipartite graph.

A final remark

Let G be a simple graph with signless Laplacian matrix Q. Then

$$\text{number of odd cycles of } G \leq \frac{\det(Q)}{4}.$$

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Thank you!

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