Linear Algebra and its Applications 560 (2019) 43–55

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

An analog of Matrix Tree Theorem for signless Laplacians

Keivan Hassani Monfared [a,1], Sudipta Mallik [b,*]

[a] Department of Mathematics and Statistics, University of Calgary, 2500 University Drive NW, Calgary, AB, T2N 1N4, Canada
[b] Department of Mathematics and Statistics, Northern Arizona University, 801 S. Osborne Dr., PO Box: 5717, Flagstaff, AZ 86011, USA

A R T I C L E   I N F O

Article history:
Received 12 May 2018
Accepted 18 September 2018
Available online 20 September 2018
Submitted by D. Stevanovic

MSC:
05C50
65F18

Keywords:
Signless Laplacian matrix
Graph
Spanning tree
Eigenvalue
Minor

A B S T R A C T

A spanning tree of a graph is a connected subgraph on all vertices with the minimum number of edges. The number of spanning trees in a graph G is given by Matrix Tree Theorem in terms of principal minors of Laplacian matrix of G. We show a similar combinatorial interpretation for principal minors of signless Laplacian Q. We also prove that the number of odd cycles in G is less than or equal to $\frac{\det(Q)}{4}$, where the equality holds if and only if G is a bipartite graph or an odd-unicyclic graph.

* Corresponding author.
E-mail addresses: k1monfared@gmail.com (K. Hassani Monfared), sudipta.mallik@nau.edu (S. Mallik).

[1] The work of this author was partially supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) and a postdoctoral fellowship of the Pacific Institute of Mathematical Sciences (PIMS).

https://doi.org/10.1016/j.laa.2018.09.016
0024-3795/© 2018 Published by Elsevier Inc.

1. Introduction

For a simple graph G on n vertices $1, 2, \ldots, n$ and m edges $1, 2, \ldots, m$ we define its *degree matrix* D, *adjacency matrix* A, and *incidence matrix* N as follows:

1. $D = [d_{ij}]$ is an $n \times n$ diagonal matrix where d_{ii} is the degree of the vertex i in G for $i = 1, 2, \ldots, n$.
2. $A = [a_{ij}]$ is an $n \times n$ matrix with zero diagonals where $a_{ij} = 1$ if vertices i and j are adjacent in G and $a_{ij} = 0$ otherwise for $i, j = 1, 2, \ldots, n$.
3. $N = [n_{ij}]$ is an $n \times m$ matrix whose rows are indexed by vertices and columns are indexed by edges of G. The entry $n_{ij} = 1$ whenever vertex i is incident with edge j (i.e., vertex i is an endpoint of edge j) and $n_{ij} = 0$ otherwise.

We define the *Laplacian matrix* L and *signless Laplacian matrix* Q to be $L = D - A$ and $Q = D + A$, respectively. It is well-known that both L and Q have nonnegative real eigenvalues [1, Sec. 1.3]. Note the relation between the spectra of L and Q:

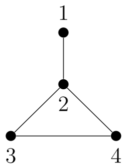
Theorem 1.1. *[1, Prop. 1.3.10] Let G be a simple graph on n vertices. Let L and Q be the Laplacian matrix and the signless Laplacian matrix of G, respectively, with eigenvalues $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ for L, and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ for Q. Then G is bipartite if and only if $\{\mu_1, \mu_2, \ldots, \mu_n\} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.*

Theorem 1.2. *[2, Prop. 2.1] The smallest eigenvalue of the signless Laplacian of a connected graph is equal to 0 if and only if the graph is bipartite. In this case 0 is a simple eigenvalue.*

We use the following notation for submatrices of an $n \times m$ matrix M: for sets $I \subset \{1, 2, \ldots, n\}$ and $J \subset \{1, 2, \ldots, m\}$,

- $M[I; J]$ denotes the submatrix of M whose rows are indexed by I and columns are indexed by J.
- $M(I; J)$ denotes the submatrix of M obtained by removing the rows indexed by I and removing the columns indexed by J.
- $M(I; J]$ denotes the submatrix of M whose columns are indexed by J, and obtained by removing rows indexed by I.

We often list the elements of I and J, separated by commas in this submatrix notation, rather than writing them as sets. For example, $M(2; 3, 7, 8]$ is a $(n - 1) \times 3$ matrix whose rows are the same as the rows of M with the second row deleted and columns are respectively the third, seventh, and eighth columns of M. Moreover, if $I = J$, we abbreviate $M(I; J)$ and $M[I; J]$ as $M(I)$ and $M[I]$ respectively. Also we abbreviate $M(\varnothing; J]$ and $M(I; \varnothing)$ as $M(; J]$ and $M(I;)$ respectively.



$$Q = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

Fig. 1. Paw G and its signless Laplacian matrix Q.

A *spanning tree* of G is a connected subgraph of G on all n vertices with minimum number of edges which is $n-1$ edges. The number of spanning trees in a graph G is denoted by $t(G)$ and is given by Matrix Tree Theorem:

Theorem 1.3 *(Matrix Tree Theorem). [1, Prop. 1.3.4] Let G be a simple graph on n vertices and L be the Laplacian matrix of G with eigenvalues $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$. Then the number $t(G)$ of spanning trees of G is*

$$t(G) = \det(L(i)) = \frac{\mu_2 \cdot \mu_3 \cdots \mu_n}{n},$$

for all $i = 1, 2, \ldots, n$.

We explore if there is an analog of the Matrix Tree Theorem for the signless Laplacian matrix Q. First note that unlike $\det(L(i))$, $\det(Q(i))$ is not necessarily the same for all i as illustrated in the following example.

Example 1.4. For the paw graph G with its signless Laplacian matrix Q in Fig. 1, $\det(Q(1)) = 7 \neq 3 = \det(Q(2)) = \det(Q(3)) = \det(Q(4))$.

The Matrix Tree Theorem can be proved by the Cauchy–Binet formula:

Theorem 1.5 *(Cauchy–Binet). [1, Prop. 1.3.5] Let $m \leq n$. For $m \times n$ matrices A and B, we have*

$$\det(AB^T) = \sum_S \det(A(;S]) \det(B(;S]),$$

where the summation runs over $\binom{n}{m}$ m-subsets S of $\{1, 2, \ldots, n\}$.

The following observation provides a decomposition of the signless Laplacian matrix Q which enables us to apply the Cauchy–Binet formula on it.

Observation 1.6. Let G be a simple graph on $n \geq 2$ vertices with m edges, and $m \geq n-1$. Suppose N and Q are the incidence matrix and signless Laplacian matrix of G, respectively. Then

(a) $Q = NN^T$,
(b) $Q(i) = N(i;)N(i;)^T$, $i = 1, 2, \dots, n$, and
(c) $\det(Q(i)) = \det(N(i;)N(i;)^T) = \sum_S \det(N(i;S])^2$, where the summation runs over all $(n-1)$-subsets S of $\{1, 2, \dots, m\}$ (by Cauchy–Binet formula 1.5).

2. Principal minors of signless Laplacians

In this section we find a combinatorial formula for a principal minor $\det(Q(i))$ for the signless Laplacian matrix Q of a given graph G. We mainly use Observation 1.6(c) given by Cauchy–Binet formula which involves determinant of submatrices of incidence matrices. This approach is completely different from the methods applied for related spectral results in [2]. But we borrow the definition of TU-subgraphs from [2] slightly modified as follows: A *TU-graph* is a graph whose connected components are trees or odd-unicyclic graphs. A *TU-subgraph* of G is a spanning subgraph of G that is a TU-graph. The following lemma finds the number of trees in a TU-graph.

Lemma 2.1. *If G is a TU-graph on n vertices with $n-k$ edges consisting of c odd-unicyclic graphs and s trees, then $s = k$.*

Proof. Suppose the number vertices of the cycles are $n_1, n_2, \dots, n_c$ and that of the trees are $t_1, t_2, \dots, t_s$. Then the total number of edges is

$$n - k = \sum_{i=1}^{c} n_i + \sum_{i=1}^{s} (t_i - 1) = n - s$$

which implies $s = k$. □

Now we find the determinant of incidence matrices of some special graphs in the following lemmas.

Lemma 2.2. *If G is an odd (resp. even) cycle, then the determinant of its incidence matrix is ± 2 (resp. zero).*

Proof. Let G be a cycle with the incidence matrix N. Then up to permutation we have

$$N = PN'Q = P \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix} Q,$$

for some permutation matrices P and Q. By a cofactor expansion across the first row we have

$$\det(N) = \det(P)\det(N')\det(Q) = (\pm 1)(1 + (-1)^{n+1})(\pm 1).$$

If n is odd (resp. even), then $\det(N) = \pm 2$ (resp. zero). □

Lemma 2.3. *If G is an odd unicyclic (resp. even unicyclic) graph, then the determinant of its incidence matrix is ± 2 (resp. 0).*

Proof. Let G be a unicyclic graph with incidence matrix N and t vertices not on the cycle. We prove the statement by induction on t. If $t = 0$, then G is an odd (resp. even) cycle and then $\det(N_i) = \pm 2$ (resp. 0) by Lemma 2.2. Assume the statement holds for some $t \geq 0$. Let G be a unicyclic graph with $t + 1$ vertices not on the cycle. Then G has a pendant vertex, say vertex i. The vertex i is incident with exactly one edge of G, say $e_l = \{i, j\}$. Then ith row of N has only one nonzero entry which is the (i, l)th entry and it is equal to 1. To find $\det(N)$ we have a cofactor expansion across the ith row and get

$$\det(N) = \pm 1 \cdot \big(\pm \det(N(i; l))\big).$$

Note that $N(i; l)$ is the incident matrix of $G(i)$, which is a unicyclic graph with t vertices not on the cycle. By induction hypothesis, $\det(N(i; l)) = \pm 2$ (resp. 0). Thus $\det(N) = \pm 1 \cdot \big(\pm \det(N(i; l))\big) = \pm 2$ (resp. 0). □

By a similar induction on the number of pendant vertices we get the following result.

Lemma 2.4. *Let H be a tree with at least one edge and N be the incidence matrix of H. Then $\det(N(i;)) = \pm 1$ for all vertices i of H.*

Lemma 2.5. *Let H be a graph on n vertices and $n - 1$ edges with incidence matrix N. If H has a connected component which is a tree and an edge which is not on the tree, then $\det(N(i;)) = 0$ for all vertices i not on the tree.*

Proof. Let H have a connected component T which is a tree and an edge e_j which is not on T. Suppose i is a vertex of G that is not on T. If T consists of just one vertex, then the corresponding row in $N(i;)$ is a zero row giving $\det(N(i;)) = 0$. Suppose T has at least two vertices. Now consider the square submatrix N' of $N(i;)$ with rows corresponding to vertices of T and columns corresponding to edges of T together with e_j. Then the column of N' corresponding to e_j is a zero row giving $\det(N') = 0$. Since entries in rows of $N_i[S]$ corresponding to T that are outside of N' are zero, the rows of $N(i;)$ corresponding to T are linearly dependent and consequently $\det(N(i;)) = 0$. □

Now we break down different scenarios that can happen to a graph with n vertices and $m = n - 1$ edges.

Proposition 2.6. *Let H be a graph on n vertices and $m = n - 1$ edges. Then one of the following is true for H.*

1. *H is a tree.*
2. *H has an even cycle and a vertex not on the cycle.*
3. *H has no even cycles, but H has a connected component with at least two odd cycles and at least two connected components which are trees.*
4. *H is a disjoint union of odd unicyclic graphs and exactly one tree, i.e., H is a TU-graph.*

Proof. If H is connected then it is a tree. This implies Case 1. Now assume H is not connected. If H has no cycles, then it is a forest with at least two connected components. This would imply that $m < n - 2$, contradicting the assumption that $m = n - 1$. Thus H has at least one cycle. Suppose H has $t \geq 2$ connected components H_i with m_i edges and n_i vertices, where the first k of them have at least a cycle and the rest are trees. For $i = 1, \ldots, k$, H_i has $m_i \geq n_i$. Note that

$$-1 = m - n = \sum_{i=1}^{t}(m_i - n_i) = \sum_{i=1}^{k}(m_i - n_i) + \sum_{i=k+1}^{t}(m_i - n_i) \tag{2.1}$$

Since H_i has a cycle for $i = 1, \ldots, k$ and H_i is a tree for $i = k + 1, \ldots, t$,

$$\ell := \sum_{i=1}^{k}(m_i - n_i) \geq 0,$$

and

$$\sum_{i=k+1}^{t}(m_i - n_i) = -(t - k).$$

Then $t - k = \ell + 1$ by (2.1). In other words, in order to make up for the extra edges in the connected components with cycles, H has to have exactly $\ell + 1$ connected components which are trees.

If H has an even cycle, then $\ell \geq 0$ and hence $t - k \geq 1$. This means there is at least one connected component which is tree and it contains a vertex which is not in the cycle. This implies Case 2. Otherwise, all of the cycles of H are odd. If it has more than one cycle in a connected component, then $\ell \geq 1$ and thus $t - k \geq 2$. This implies Case 3. Otherwise, each H_i with $i = 1, \ldots, k$ has exactly one cycle in it, which implies $\ell = 0$, and then $t - k = 1$. This implies Case 4. □

Theorem 2.7. *Let G be a simple connected graph on $n \geq 2$ vertices and m edges with the incidence matrix N. Let i be an integer from $\{1, 2, \ldots, n\}$. Let S be an $(n - 1)$-subset of*

$\{1, 2, \ldots, m\}$ and H be a spanning subgraph of G with edges indexed by S. Then one of the following holds for H.

1. *H is a tree. Then* $\det(N(i; S]) = \pm 1$.
2. *H has an even cycle and a vertex not on the cycle. Then* $\det(N(i; S]) = 0$.
3. *H has no even cycles, but it has a connected component with at least two odd cycles and at least two connected components which are trees. Then* $\det(N(i; S]) = 0$.
4. *H is a TU-subgraph of G consisting of c odd-unicyclic graphs $U_1, U_2, \ldots, U_c$ and a unique tree T. If i is a vertex of U_j for some $j = 1, 2, \ldots, c$, then* $\det(N(i; S]) = 0$. *If i is a vertex of T, then* $\det(N(i; S]) = \pm 2^c$.

Proof. Suppose vertices and edges of G are $1, 2, \ldots, n$ and $e_1, e_2, \ldots, e_m$, respectively. Note that $m \geq n - 1$ since G is connected.

1. Suppose H is a tree. Since $n \geq 2$, H has an edge. Then by Lemma 2.4, $\det(N(i; S]) = \pm 1$.
2. Suppose H contains an even cycle C as a subgraph and a vertex j not on C.

   Case 1. Vertex i is not in C.
   Then the square submatrix N' of $N(i; S]$ corresponding to C has determinant zero by Lemma 2.3. Since entries in columns of $N(i; S]$ corresponding to C that are outside of N' are zero, the columns of $N(i; S]$ corresponding to C are linearly dependent and consequently $\det(N(i; S]) = 0$.

   Case 2. Vertex i is in C.
   Since i is in C, we have $j \neq i$. Consider the square submatrix N' of $N(i; S]$ that has rows corresponding to vertex j and vertices of C excluding i and columns corresponding to edges of C. Since vertex j is not on C, the row of N' corresponding to vertex j is a zero row and consequently $\det(N') = 0$. Since entries in columns of $N_i[S]$ corresponding to C that are outside of N' are zero, the columns of $N(i; S]$ corresponding to C are linearly dependent and consequently $\det(N(i; S]) = 0$.
3. Suppose H has no even cycles, but it has a connected component with at least two odd cycles and at least two connected components which are trees. Then vertex i is not in one of the trees. Then $\det(N(i; S]) = 0$ by Lemma 2.5.
4. Suppose H is a TU-subgraph of G consisting of c odd-unicyclic graphs $U_1, U_2, \ldots, U_c$ and a unique tree T. If i is a vertex of U_j for some $j = 1, \ldots, c$, then $\det(N(i; S]) = 0$ by Lemma 2.5. If i is a vertex of the tree T, then $N(i; S]$ is a direct sum of incidence matrices of odd-unicyclic graphs $U_1, U_2, \ldots, U_c$ and the incidence matrix of the tree T with one row deleted (which does not exist when T is a tree on the single vertex i). By Lemma 2.3 and 2.4, $\det(N(i; S]) = (\pm 2)^c \cdot (\pm 1) = \pm 2^c$. □

The preceding results are summarized in the following theorem.

Theorem 2.8. *Let G be a simple connected graph on $n \geq 2$ vertices and m edges with the incidence matrix N. Let i be an integer from $\{1, 2, \dots, n\}$. Let S be an $(n-1)$-subset of $\{1, 2, \dots, m\}$ and H be a spanning subgraph of G with edges indexed by S.*

(a) If H is not a TU-subgraph of G, then $\det(N(i; S]) = 0$.
(b) Suppose H is a TU-subgraph of G consisting of c odd-unicyclic graphs $U_1, U_2, \dots, U_c$ and a unique tree T. If i is a vertex of U_j for some $j = 1, 2, \dots, c$, then $\det(N(i; S]) = 0$. If i is a vertex of T, then $\det(N(i; S]) = \pm 2^c$.

For a TU-subgraph H of G, the number of connected components that are odd-unicyclic graphs is denoted by $c(H)$. So a TU-subgraph H on $n-1$ edges with $c(H) = 0$ is a spanning tree of G.

Theorem 2.9. *Let G be a simple connected graph on $n \geq 2$ vertices $1, 2, \dots, n$ with the signless Laplacian matrix Q. Then*

$$\det(Q(i)) = \sum_{H} 4^{c(H)},$$

where the summation runs over all TU-subgraphs H of G with $n-1$ edges consisting of a unique tree on vertex i and $c(H)$ odd-unicyclic graphs.

Proof. By Observation 1.6, we have,

$$\det(Q(i)) = \sum_{S} \det(N(i; S])^2,$$

where the summation runs over all $(n-1)$-subsets S of $\{1, 2, \dots, m\}$. By Theorem 2.8, we have,

$$\det(Q(i)) = \sum_{S} \det(N(i; S])^2 = \sum_{H} (\pm 2^{c(H)})^2 = \sum_{H} 4^{c(H)},$$

where the summation runs over all TU-subgraphs H of G with $n-1$ edges consisting of a unique tree on vertex i and $c(H)$ odd-unicyclic graphs. □

Example 2.10. Consider the Paw G and its signless Laplacian matrix Q in Fig. 1. To determine $\det(Q(1))$, consider the TU-subgraphs of G with 3 edges consisting of a unique tree on vertex 1: H_1, H_2, H_3, H_4 in Fig. 2. Note $c(H_1) = c(H_2) = c(H_3) = 0$ and $c(H_4) = 1$. Then by Theorem 2.9,

$$\det(Q(1)) = \sum_{H} 4^{c(H)} = 4^{c(H_1)} + 4^{c(H_2)} + 4^{c(H_3)} + 4^{c(H_4)} = 4^0 + 4^0 + 4^0 + 4^1 = 7.$$

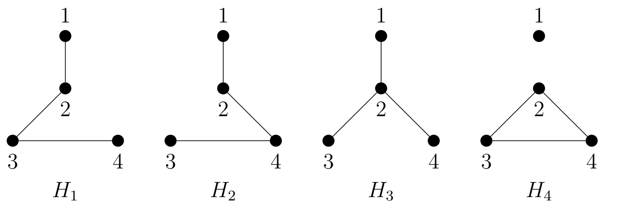


Fig. 2. TU-subgraphs of Paw G with 3 edges consisting of a unique tree on vertex 1.

Corollary 2.11. *Let G be a simple connected graph on $n \geq 2$ vertices $1, 2, \ldots, n$. Let Q be the signless Laplacian matrix of G with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then*

(a) $\det(Q(i)) \geq t(G)$, *the number of spanning trees of G, where the equality holds if and only if all odd cycles of G contain vertex i.*

(b)

$$\frac{1}{n} \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_{n-1}} = \frac{1}{n} \sum_{i=1}^{n} \det(Q(i)) \geq t(G),$$

where the equality holds if and only if G is an odd cycle or a bipartite graph.

Proof. (a) First note that a TU-subgraph H on $n-1$ edges with $c(H) = 0$ is a spanning tree of G. Then $\det(Q(i)) = \sum_H 4^{c(H)} \geq \sum_T 4^0$, where the sum runs over all spanning trees T of G containing vertex i. So $\det(Q(i))$ is greater than or equal to the number of spanning trees of G containing vertex i. Since each spanning tree contains vertex i, $\det(Q(i)) \geq t(G)$ where the equality holds if and only if all odd-unicyclic subgraphs of G contain vertex i by Theorem 2.9. Finally note that all odd-unicyclic subgraphs of G contain vertex i if and only if all odd cycles of G contain vertex i.

(b) The first equality follows from the well-known linear algebraic result

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_{n-1}} = \sum_{i=1}^{n} \det(Q(i)).$$

Now by (a) $\det(Q(i)) \geq t(G)$ where the equality holds if and only if all odd cycles of G contain vertex i. Then

$$\frac{1}{n} \sum_{i=1}^{n} \det(Q(i)) \geq t(G)$$

where the equality holds if and only if $\det(Q(i)) = t(G)$ for all $i = 1, 2, \ldots, n$. So the equality holds if and only if all odd cycles of G contain every vertex of G which means G is an odd cycle or a bipartite graph (G has no odd cycles). □

3. Number of odd cycles in a graph

In this section we find a combinatorial formula for $\det(Q)$ for the signless Laplacian matrix Q of a given graph G. As a corollary we show that the number of odd cycles in G is less than or equal to $\frac{\det(Q)}{4}$.

Proposition 3.1. *Let H be a graph on n vertices and $m = n$ edges. Then one of the following is true for H.*

1. *H has a connected component which is a tree.*
2. *All connected components of H are unicyclic and at least one of them is even-unicyclic.*
3. *All connected components of H are odd-unicyclic.*

Proof. Suppose H has $t \geq 2$ connected components H_i with m_i edges and n_i vertices, where the first k of them have at least one cycle and the rest are trees. For $i = 1, \dots, k$, H_i has $m_i \geq n_i$. Note that

$$0 = m - n = \sum_{i=1}^{t}(m_i - n_i) = \sum_{i=1}^{k}(m_i - n_i) + \sum_{i=k+1}^{t}(m_i - n_i) \tag{3.1}$$

Since H_i has a cycle for $i = 1, \dots, k$ and H_i is a tree for $i = k+1, \dots, t$,

$$\ell := \sum_{i=1}^{k}(m_i - n_i) \geq 0,$$

and

$$\sum_{i=k+1}^{t}(m_i - n_i) = -(t-k).$$

Then $t - k = \ell$ by (3.1). If H has a connected component which is a tree, we have Case 1. Otherwise $t - k = 0$ which implies $\ell = \sum_{i=1}^{k}(m_i - n_i) = 0$. Then $m_i = n_i$, for $i = 1, 2, \dots, k$, i.e., all connected components of H are unicyclic. If one of the unicyclic components is even-unicyclic, we get Case 2. Otherwise all connected components of H are odd-unicyclic which is Case 3. Finally if H is connected, it is unicyclic and consequently it is Case 2 or 3. □

Lemma 3.2. *Let H be a graph on n vertices and n edges with incidence matrix N. If H has a connected component which is a tree and an edge which is not on the tree, then $\det(N) = 0$.*

Proof. Let H have a connected component T which is a tree and an edge e_j which is not on T. If T consists of just one vertex, say i, then row i of N is a zero row giving $\det(N) = 0$. Suppose T has at least two vertices. Now consider the square submatrix N' of N with rows corresponding to vertices of T and columns corresponding to edges of T together with e_j. Then the column of N' corresponding to e_j is a zero row giving $\det(N') = 0$. Since entries in rows of N corresponding to T that are outside of N' are zero, the rows of N corresponding to T are linearly dependent and consequently $\det(N) = 0$. □

Theorem 3.3. *Let G be a simple graph on n vertices and $m \geq n$ edges with the incidence matrix N. Let S be a n-subset of $\{1, 2, \ldots, m\}$ and H be a spanning subgraph of G with edges indexed by S. Then one of the following is true for H:*

1. *H has a connected component which is a tree. Then* $\det(N[S]) = 0$.
2. *All connected components of H are unicyclic and at least one of them is even-unicyclic. Then* $\det(N[S]) = 0$.
3. *H has k connected components which are all odd-unicyclic. Then* $\det(N[S]) = \pm 2^k$.

Proof. 1. Suppose H has a connected component which is a tree. Since H has n edges, H has an edge not on the tree. Then $\det(N[S]) = 0$ by Lemma 3.2.
2. Suppose all connected components of H are unicyclic and at least one of them is even-unicyclic. Since $N[S]$ is a direct sum of incidence matrices of unicyclic graphs where at least one of them is even-unicyclic, then $\det(N[S]) = 0$ by Lemma 2.2.
3. Suppose H has k connected components which are all odd-unicyclic. Since $N[S]$ is a direct sum of incidence matrices of k odd-unicyclic graphs, then $\det(N[S]) = (\pm 2)^k = \pm 2^k$ by Lemma 2.2. □

By Theorem 1.5 and 3.3, we have the following theorem.

Theorem 3.4. *Let G be a simple graph on n vertices with signless Laplacian matrix Q. Then*

$$\det(Q) = \sum_{H} 4^{c(H)},$$

where the summation runs over all spanning subgraphs H of G on n edges whose all connected components are odd-unicyclic.

Proof. By Theorem 1.5 and Observation 1.6,

$$\det(Q) = \det(NN^T) = \sum_{S} \det(N(;S])^2,$$

where the summation runs over all n-subsets S of $\{1, 2, \ldots, m\}$. By Theorem 3.3, we have

$$\det(Q) = \sum_S \det(N(;S])^2 = \sum_H (\pm 2^{c(H)})^2 = \sum_H 4^{c(H)},$$

where the summation runs over all spanning subgraphs H of G whose all connected components are odd-unicyclic. $\square$

Let $ous(G)$ denote the number of spanning subgraphs H of a graph G where each connected component of H is an odd-unicyclic graph. So $ous(G)$ is the number of TU-subgraphs of G whose all connected components are odd-unicyclic. Note that $c(H) \geq 1$ for all spanning subgraphs H of G whose all connected components are odd-unicyclic. By Theorem 3.4, we have an upper bound for $ous(G)$.

Corollary 3.5. *Let G be a simple graph with signless Laplacian matrix Q. Then* $\det(Q) \geq 4ous(G)$.

For example, if G is bipartite graph, then $\frac{\det(Q)}{4} = 0 = ous(G)$. If G is an odd-unicyclic graph, then $\frac{\det(Q)}{4} = 1 = ous(G)$.

Note that by appending edges to an odd cycle in G we get at least one TU-subgraph of G with a unique odd-unicyclic connected component. Let $oc(G)$ denote the number of odd cycles in a graph G. Then $oc(G) \leq ous(G)$, where the equality holds if and only if G is a bipartite graph or an odd-unicyclic graph. Then we have the following corollary.

Corollary 3.6. *Let G be a simple graph with signless Laplacian matrix Q. Then $\frac{\det(Q)}{4} \geq oc(G)$, the number of odd cycles in G, where the equality holds if and only if G is a bipartite graph or an odd-unicyclic graph.*

4. Open problems

In this section we pose some problems related to results in Sections 2 and 3. First recall Corollary 3.6 which gives a linear algebraic sharp upper bound for the number of odd cycles in a graph. So an immediate question would be the following:

Question 4.1. Find a linear algebraic (sharp) upper bound of the number of even cycles in a simple graph.

To answer this one may like to apply Cauchy–Binet Theorem as done in Sections 2 and 3. Then a special $n \times m$ matrix R will be required with the following properties:

1. RR^T is a decomposition of a fixed matrix for a given graph G.
2. If G is an even (resp. odd) cycle, then $\det(R)$ is $\pm c$ (resp. zero) for some fixed nonzero number c.

For other open questions consider a simple connected graph G on n vertices and $m \geq n$ edges with signless Laplacian matrix Q. The characteristic polynomial of Q is

$$P_Q(x) = \det(xI_n - Q) = x^n + \sum_{i=1}^{n} a_i x^{n-i}.$$

It is not hard to see that $a_1 = -2m$ and $a_2 = 2m^2 - m - \frac{1}{2}\sum_{i=1}^{n} d_i^2$ where $(d_1, d_2, \ldots, d_n)$ is the degree-sequence of G. Theorem 4.4 in [2] provides a broad combinatorial interpretation for a_i, $i = 1, 2, \ldots, n$. A combinatorial expression for a_3 is obtained in [3, Thm. 2.6] by using mainly Theorem 4.4 in [2]. Note that

$$a_3 = (-1)^3 \sum_{1 \le i_1 < i_2 < i_3 \le n} \det(Q[i_1, i_2, i_3]).$$

So it may not be difficult to find corresponding combinatorial interpretation of $\det(Q[i_1, i_2, i_3])$ in terms of subgraphs on three edges. Similarly we can investigate other coefficients and corresponding minors which we essentially did for a_n and a_{n-1} in Sections 3 and 2 respectively. So the next coefficient to study is a_{n-2} which entails the following question:

Question 4.2. Find a combinatorial expression or a lower bound for $\det(Q(i_1, i_2))$.

By Cauchy–Binet Theorem,

$$\det(Q(i_1, i_2)) = \sum_{S} \det(N(i_1, i_2; S])^2,$$

where the summation runs over all $(n-2)$-subsets S of the edge set $\{1, 2, \ldots, m\}$. So it comes down to finding a combinatorial interpretation of $\det(N(i_1, i_2; S])$.

References

[1] Andries E. Brouwer, Williem H. Haemers, Spectra of Graphs, Springer-Verlag, New York, 2012.
[2] Dragoš Cvetković, Peter Rowlinson, Slobodan K. Simić, Signless Laplacians of finite graphs, Linear Algebra Appl. 423 (2007) 155–171.
[3] Jianfeng Wang, Qiongxiang Huang, Xinhui An, Francesco Belardo, Some results on the signless Laplacians of graphs, Appl. Math. Lett. 23 (2010) 1045–1049.