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The nowhere-zero eigenbasis problem for a graph

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ABSTRACT

Using the implicit function theorem it is shown that for any n distinct real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$, and for each connected graph G of order n, there is a real symmetric matrix A whose graph is G, the eigenvalues of A are $\lambda_1, \dots, \lambda_n$, and every entry in each eigenvector of A is nonzero.

1. Introduction

The *graph of* an $n \times n$ real symmetric matrix $A = [a_{ij}]$ is the (simple) graph G on n vertices $1, 2, \dots, n$ with edges $\{i, j\}$ if and only if $a_{ij} \neq 0$ and $i \neq j$. In the recent years considerable research has concerned the relationship between the spectrum of a symmetric matrix and its graph (for example see [3] and the references therein). The

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first result we recall asserts that every graph realizes each spectrum consisting of distinct eigenvalues. We denote the multi-set of eigenvalues of A by spec(A).

Theorem 1.1. *[3, Theorem 2.2.1] Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a set of n distinct real numbers and G be a graph on n vertices. Then there is a real symmetric matrix A whose graph is G and spec$(A) = \Lambda$.*

The nowhere-zero eigenbasis problem for G, raised by Shaun Fallat [2], is an extension of the Theorem 1.1 that puts extra requirements on the matrix A, namely that none of its eigenvectors has a zero entry. Note that if G is not connected, then A will be a direct sum of matrices and hence its eigenvectors will have zero entries. Thus, it is necessary to assume G is connected. More formally, the problem we study in this paper is the following.

> **The nowhere-zero eigenbasis problem for G.** For a given connected graph G on n vertices and given list $\lambda_1, \lambda_2, \ldots, \lambda_n$, of n distinct real numbers, does there exist a real symmetric matrix A whose graph is G, its eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n$, and none of the eigenvectors of A has a zero entry?

For a square matrix A and subsets α and β of indices, $A[\alpha, \beta]$ is the submatrix of A with rows indexed by α and columns indexed by β. The matrix obtained from A by deleting its j-th row and j-th column is denoted by $A(j)$. Note that if the j-th entry of an eigenvector of A is zero, then A and $A(j)$ share the eigenvalue corresponding to that eigenvector. The converse is also true; namely, if A and $A(j)$ share an eigenvalue λ, then there is an eigenvector of A corresponding to λ whose j-th entry is zero. To see this, assume $j = 1$ and note that if $\boldsymbol{x}$ and $\boldsymbol{y}$ are eigenvectors of $A(1)$ and A, respectively, corresponding to the eigenvalue λ, then either the first entry of $\boldsymbol{x}$ is zero, or $A[\{2, \ldots, n\}, \{1\}]$ is in the column space of $A(1) - \lambda I$, which along with the symmetry of A imply that

$$\begin{bmatrix} 0 \\ \boldsymbol{y} \end{bmatrix}$$

is an eigenvector of A corresponding to λ. From this perspective, the nowhere-zero eigenbasis problem concerns the existence of a real symmetric matrix A with prescribed spectrum and graph such that $\sigma(A) \cap \sigma(A(j)) = \emptyset$ for each j.

The *Cauchy interlacing inequalities* guarantee that $\sigma(A(j))$ interlaces $\sigma(A)$, that is, if $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are eigenvalues of A and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$ are eigenvalues of $A(j)$, then

$$\lambda_i \leq \mu_i \leq \lambda_{i+1}, \text{ for } i = 1, 2, \ldots, n-1. \tag{1}$$

Thus the condition $\sigma(A) \cap \sigma(A(j)) = \emptyset$ is equivalent to the condition that $\sigma(A(j))$ strictly interlaces $\sigma(A)$, that is, all the inequalities in (1) are strict. Hence, from this perspective,

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 4 \end{bmatrix}$$

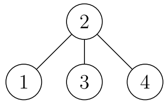

Fig. 1. Matrix A whose graph is a star on 4 vertices.

the nowhere-zero eigenbasis problem concerns finding a matrix of order n with prescribed spectrum and graph such that there are strict inequalities in the interlacing inequalities for each principle submatrix of order $n - 1$.

We now recall a useful concept from [3]. Let A be a real symmetric matrix whose graph is a tree T. For a vertex v of T, we let $T(v)$ denote the graph obtained from T by deleting vertex v. As T is a tree, the connected components of $T(v)$ will be trees, and these are called *the branches of* T *at* v. If $w \sim_T v$, then the branch of T at v that contains w is denoted by $T_w(v)$, and the principal submatrix of A determined by the vertices of $T_w(v)$ is denoted $A[T_w(v)]$.

Definition 1.2. Let A be a real symmetric matrix or order n whose graph is a tree T. A vertex v of T is a *Duarte vertex of* A if

(i) $n = 1$ and v is the only vertex of T, or
(ii) $n > 1$, the eigenvalues of $A(v)$ strictly interlace those of A, and w is a Duarte vertex for $A[T_w(v)]$, for each vertex w in $T(v)$ that is adjacent to v in T.

When v is a Duarte vertex of A, we say that A has the Duarte property with respect to v. We note that if A has the Duarte property with respect to a vertex v, then none of its eigenvectors has a zero entry in row v. It is shown that [3, Remark 3.1.3] for a real symmetric matrix A whose graph is a tree, a vertex v is a Duarte vertex for A if and only if $\sigma(A(v)) \cap \sigma(A) = \emptyset$. Thus, if the graph of A is a tree, then no eigenvector of A has a 0 if and only if A has the Duarte property with respect to each vertex.

Example 1.3 below suggests that the resolution of the nowhere-zero eigenbasis problem is perhaps not easy, as not every matrix with the Duarte property with respect to a vertex has a the Duarte property with respect to all the vertices.

Example 1.3. Consider the following matrix A whose graph is a star on 4 vertices shown in Fig. 1. The eigenvalues of A are approximately 2, -0.164, 2.773, and 4.391. The eigenvalues of $A(1)$ are approximately 0.186, 2.471, and 4.343 which strictly interlace the spectrum of A, and hence A has the Duarte property with respect to vertex 1. But the eigenvalues of $A(2)$ are 2, 2 and 4. That means A does not have the Duarte property with respect to vertex 2.

In Section 2 we provide several preliminary results regarding the Duarte property. In Section 3 we provide a solution for the nowhere-zero eigenbasis problem, using the implicit function theorem, in the case that the graph is a tree. In Section 4 we provide a solution for the nowhere-zero eigenbasis problem for connected graphs, again using the implicit function theorem. Finally, in the last section we use the results of Section 4 to show that for every pair of connected graphs G and H with the same number of vertices there is a symmetric matrix with two distinct eigenvalues whose graph is the join of G and H.

2. Preliminary results

Let K be a subset of vertices of T. The induced subgraph of T on vertices in K is denoted by $T[K]$, and the matrix obtained from A by keeping the vertices indexed by K is denoted by $A[K]$. In this section we study several properties implied by the Duarte property.

Lemma 2.1. *Let A be a real symmetric matrix whose graph is a tree T. Then vertex v of T is not a Duarte vertex for A if and only if there is a vertex u and a branch K of T at u that does not contain v such that $A[K]$ and $A[K \cup \{u\}]$ have a common eigenvalue.*

Proof. First assume $v = v_0$ is not a Duarte vertex for A. We prove the existence of such u and K by induction on the number of vertices. Since v is not Duarte, T has at least two vertices. If $A(v)$ and A have a common eigenvalue, then we can take $u = v$ and K to be a branch of T at v having that common eigenvalue. Otherwise, there is a branch L of v such that $A[L]$ doesn't have the Duarte Property with respect to the neighbor w of v that is in L. By the induction hypothesis, there exists vertex u and a branch of K at u in L that does not contain w with $(A[L])[K]$ and $(A[L])[K \cup \{u\}]$ having a common eigenvalue. Then u and K satisfy the desired properties for T.

Conversely, assume that there is a vertex u and a connected component K of $T \setminus \{u\}$ not containing v, such that $A[K]$ and $A[K \cup \{u\}]$ have a common eigenvalue. Then the path $v = v_0 \sim_T v_1 \sim_T \cdots \sim_T v_k = u$ from v to u is disjoint from K. Let $T' = T_{v_k}(\{v_0, v_1, \ldots, v_{k-1}\})$. Then $A'(v_k)$ has a common eigenvalue with A', and v is not a Duarte vertex for A. $\square$

The next result was proven in [5, Corollary 1.3.2] for the eigenvalue 0. The result for an arbitrary eigenvalue follows by replacing A by $A - \lambda I$. Let $m_A(\lambda)$ denote the multiplicity of an eigenvalue λ of A.

Lemma 2.2. *Let A be a real symmetric matrix whose graph is a tree T. If A and $A(v)$ have a common eigenvalue λ, then there is a vertex u of T and two connected components K and L of $T \setminus \{u\}$ not containing v, where $A[K]$ and $A[L]$ each have an eigenvalue λ and $m_{A(u)}(\lambda) = m_A(\lambda) + 1$.*

Lemmas 2.1 and 2.2 together imply the following.

Corollary 2.3. *Let A be a real symmetric matrix whose graph is a tree T. If a vertex v of T is not a Duarte vertex for A, then there is a vertex u and two connected components K and L of $T \setminus \{u\}$ not containing v such that $A[K]$ and $A[L]$ have a common eigenvalue.*

Proof. If v is not a Duarte vertex for A, then by Lemma 2.1 there is a vertex w and a connected component M of $T(w)$ not containing v such that $A[M]$ and $A[M \cup \{w\}]$ have a common eigenvalue μ. By Lemma 2.2 there is a vertex u of $T[M \cup \{w\}]$ and two connected components K and L of $T[M \cup \{w\}](u)$ not containing w such that $A[K]$ and $A[L]$ have a common eigenvalue μ. $\square$

Now we show that if a vertex is a Duarte vertex for a matrix, then it is a Duarte vertex for the principal submatrix obtained by deleting some of the connected components adjacent to that vertex.

Lemma 2.4. *Let A be a real symmetric matrix whose graph is a tree T. Let v be a vertex of T and $K_1, K_2, \ldots, K_\ell$ be the connected components of $T \setminus \{v\}$, and set*

$$T' = T \setminus \bigcup_{i \in I} K_i,$$

for some $I \subseteq \{1, 2, \ldots, \ell\}$. If v is a Duarte vertex for A, then v is also a Duarte vertex for $A[T']$.

Proof. We first show that for K, a connected component of $T \setminus \{v\}$, if v is a Duarte vertex for A, then v is also a Duarte vertex for $A(K)$. Let $L = T \setminus K$, and $B = A(K) = A[L]$. We want to show that v is a Duarte vertex for B. If v is not a Duarte vertex for B, then by Corollary 2.3 there is a vertex u of L and two connected components M and N of $L \setminus \{u\}$ such that $B[M]$ and $B[N]$ have a common eigenvalue. This leads to the contradiction that $A[M]$ and $A[N]$ have a common eigenvalue. Hence v is a Duarte vertex for B.

Repeating this process shows that if v is a Duarte vertex for A then it is a Duarte vertex for

$$A(\bigcup_{i \in I} K_i). \quad \square$$

Furthermore, Lemma 2.4 implies the following technical corollary.

Corollary 2.5. *Let A be a real symmetric matrix whose graph is a tree T. Let v_0 be a vertex of T, with neighbors $v_1, v_2, \dots, v_k$. Let $T_1, T_2, \dots, T_k$ be the connected components of $T \setminus \{v_0\}$, where each T_i contains v_i for $i = 1, 2, \dots, k$. If v_1 is a Duarte vertex for A, then each v_i is a Duarte vertex for $A[T_i]$, for $i = 1, 2, \dots, k$.*

Proof. The fact that $v_2, v_3, \dots, v_k$ are Duarte vertices for $A[T_2], A[T_3], \dots, A[T_k]$, respectively, follows from the definition of the Duarte property of v_1 for A. Also Lemma 2.4 implies v_1 is a Duarte vertex for $A[T_1]$ by choosing $v = v_1$ and $K = T_{v_0}(v_1)$. $\square$

Any matrix with the Duarte property with respect to a vertex v has distinct eigenvalues. Hence Corollary 2.5 implies the following.

Corollary 2.6. *Let A be a real symmetric matrix whose graph is a tree T. Let v_0 be a vertex of T, with neighbors $v_1, v_2, \dots, v_k$. Let $T_1, T_2, \dots, T_k$ be the connected components of $T \setminus \{v_0\}$, where each T_i contains v_i for $i = 1, 2, \dots, k$. If v_1 is a Duarte vertex for A, then each $A[T_i]$ has distinct eigenvalues, for $i = 1, 2, \dots, k$.*

3. The Jacobian method and trees

In this section we will construct a matrix whose graph is a given tree T on n vertices $1, 2, \dots, n$, its spectrum is a prescribed set of n distinct real numbers $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$, and none of its eigenvectors have a zero entry. Informally, the idea is the following. Let $\mathcal{M} = \{\mu_1, \mu_2, \dots, \mu_{n-1}\}$ so that $\mathcal{M}$ strictly interlaces Λ, that is,

$$\lambda_i < \mu_i < \lambda_{i+1}$$

for $i = 1, 2, \dots, n-1$. By [4, Theorem 4.2] there is a matrix A with graph T and spectrum Λ such that the spectrum of $A(1)$ is $\mathcal{M}$. Since $\mathcal{M}$ strictly interlaces Λ, the matrix A has the Duarte property with respect to vertex 1. If A has the Duarte property with respect to each vertex we are done. Otherwise we use the implicit function theorem to show that the matrix can be perturbed so that its graph and spectrum remain the same while the number of vertices that are not Duarte vertices decreases. This process is repeated until we reach a matrix with the same graph and spectrum as A, and having the Duarte property with respect to each vertex. Consequently, none of the eigenvectors of this matrix has a zero entry.

Let T be a tree on n vertices, and v be a fixed vertex of T with neighbors $v_1, v_2, \dots, v_k$, and let $T_i = T_{v_i}(v)$. Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be a set of n distinct real numbers. By [4, Theorem 4.2] there is a matrix A whose graph is T, its spectrum is Λ, and v_1 is a Duarte vertex for A. Without loss of generality assume A has the following form

$$A = \left[\begin{array}{c|c|c|c|c} a_{vv} & \boldsymbol{a}_1^T & \boldsymbol{a}_2^T & \cdots & \boldsymbol{a}_k^T \\ \hline \boldsymbol{a}_1 & A_1 & O & \cdots & O \\ \hline \boldsymbol{a}_2 & O & A_2 & \cdots & O \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline \boldsymbol{a}_k & O & O & \cdots & A_k \end{array}\right], \tag{2}$$

where the first row and column correspond to vertex v, only the first entry of each $\boldsymbol{a}_\ell^T$ is nonzero, and each $A_i = A[T_i]$. Furthermore assume that A_1 is $m \times m$.

Let B be the matrix obtained by replacing each entry of A outside of A_1 by 0, that is,

$$B = \left[\begin{array}{c|c|c|c|c} 0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline \mathbf{0} & A_1 & O & \cdots & O \\ \hline \mathbf{0} & O & O & \cdots & O \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{0} & O & O & \cdots & O \end{array}\right]. \tag{3}$$

First, we prove the following technical lemma. Here $[\cdot,\cdot]$ denotes the commutator operator.

Lemma 3.1. *Let A be an $n \times n$ matrix with distinct eigenvalues, such that A and B have the form (2) and (3), respectively. If A has the Duarte property with respect to vertex v,* *and if $q(x)$ is a polynomial of degree less than m such that $[A, q(B)] = O$, then $q(x)$ is the zero polynomial.*

Proof. Assume that A has the Duarte property with respect to v and $q(x)$ is a polynomial of degree less than m such that $[A, q(B)] = O$. By Corollary 2.6, the eigenvalues of A_1 are distinct. Let

$$A_1 = \sum_{\ell=1}^{m} \mu_\ell \boldsymbol{w}_\ell \boldsymbol{w}_\ell^T$$

be the spectral decomposition of A_1. Then

$$q(A_1) = \sum_{\ell=1}^{m} q(\mu_\ell) \boldsymbol{w}_\ell \boldsymbol{w}_\ell^T.$$

First note that since $[A, q(B)] = O$, the form of B implies that

$$\boldsymbol{e}^T q(A_1) = \mathbf{0}^T, \tag{4}$$

where $\boldsymbol{e}$ is the standard unit vector of appropriate size with 1 in the first position and zeros elsewhere. Also, since A has the Duarte property with respect to vertex v_1, by Corollary 2.5 A_1 has the Duarte property with respect to vertex v_1. This implies that $\boldsymbol{e}^T \boldsymbol{w}_\ell \neq 0$ for $\ell = 1, 2, \ldots, m$. From (4) we have:

$$\boldsymbol{e}^T q(A_1) = \sum_{\ell=1}^{m} q(\mu_\ell) \boldsymbol{e}^T \boldsymbol{w}_\ell \boldsymbol{w}_\ell^T = \mathbf{0}^T.$$

Since $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$ is linearly independent, each $q(\mu_\ell)\boldsymbol{e}^T \boldsymbol{w}_\ell = 0$. And since $\boldsymbol{e}^T w_\ell \neq 0$ we have $q(\mu_\ell) = 0$, for $\ell = 1, 2, \ldots, m$. Finally, since $\mu_1, \mu_2, \ldots, \mu_m$ are distinct and $\deg(q) < m$, we conclude that $q(x)$ is the zero polynomial. $\square$

We now follow the method introduced in [3], known as the Jacobian method. Let M be the symmetric matrix obtained from A by replacing its diagonal entries by $2x_1, 2x_2, \ldots, 2x_n$ and its nonzero off-diagonal entries by $x_{n+1}, x_{n+2}, \ldots, x_{2n-1}$. Set

$$N = \left[\begin{array}{c|c|c|c|c} 0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline \mathbf{0} & M_1 & O & \cdots & O \\ \hline \mathbf{0} & O & O & \cdots & O \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{0} & O & O & \cdots & O \end{array}\right].$$

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_{2n-1})$. Define $f : \mathbb{R}^{2n-1} \to \mathbb{R}^{n+m}$ by

$$f(\boldsymbol{x}) = \left(\frac{\operatorname{tr} M}{2}, \frac{\operatorname{tr} M^2}{4}, \ldots, \frac{\operatorname{tr} M^n}{2n}, \frac{\operatorname{tr} N}{2}, \frac{\operatorname{tr} N^2}{4}, \ldots, \frac{\operatorname{tr} N^m}{2m}\right). \tag{5}$$

Note that

$$\operatorname{Jac}(f) = \left[\begin{array}{c} \dfrac{\partial \operatorname{tr} M^i}{\partial x_j} \\ \hline \dfrac{\partial \operatorname{tr} N^{i-n}}{\partial x_j} \end{array}\right]$$

is an $(n+m) \times (2n-1)$ matrix. Let $J = \operatorname{Jac}(f)\big|_A$; that is, J is the matrix obtained from $\operatorname{Jac}(f)$ by setting x_i to be the corresponding entry of A. We now show that Lemma 3.1 implies that the rows of J are linearly independent.

Lemma 3.2. *Let f be defined by (5), and let A be as in (2) such that A has the Duarte property with respect to vertex v_1. Then rows of $J = \operatorname{Jac}(f)\big|_A$ are linearly independent.*

Proof. By [4, Lemma 3.1] we have

$$J = \left[\begin{array}{ccc|ccc} I_{11} & \cdots & I_{nn} & I_{i_1j_1} & \cdots & I_{i_{n-1}j_{n-1}} \\ A_{11} & \cdots & A_{nn} & A_{i_1j_1} & \cdots & A_{i_{n-1}j_{n-1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{11}^{n-1} & \cdots & A_{nn}^{n-1} & A_{i_1j_1}^{n-1} & \cdots & A_{i_{n-1}j_{n-1}}^{n-1} \\ \hline I'_{11} & \cdots & I'_{nn} & I'_{i_1j_1} & \cdots & I'_{i_{n-1}j_{n-1}} \\ B_{11} & \cdots & B_{nn} & B_{i_1j_1} & \cdots & B_{i_{n-1}j_{n-1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B_{11}^{m-1} & \cdots & B_{nn}^{m-1} & B_{i_1j_1}^{m-1} & \cdots & B_{i_{n-1}j_{n-1}}^{m-1} \end{array}\right],$$

where I' denotes the matrix obtained form the identity matrix by replacing all entries outside the same block as A_1 with 0, and each $i_\ell j_\ell$ denotes a nonzero position of A. Let

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\alpha_0, \alpha_1, \dots, \alpha_{n-1}, \beta_0, \beta_1, \dots, \beta_{m-1}).$$

Assume $(\boldsymbol{\alpha}, \boldsymbol{\beta})J = \mathbf{0}$. It suffices to show that $\boldsymbol{\alpha} = \mathbf{0}$ and $\boldsymbol{\beta} = \mathbf{0}$. Let

$$\begin{aligned} p(x) &= \alpha_0 x^0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1}, \\ q(x) &= \beta_0 x^0 + \beta_1 x + \cdots + \beta_{m-1} x^{m-1} \end{aligned}$$

and $X = p(A) + q(B)$. Note that rows of J are linearly independent if and only if both $p(x)$ and $q(x)$ are the zero polynomial.

Let $\circ$ denote the Schur (entry-wise) product of two matrices. From $(\boldsymbol{\alpha}, \boldsymbol{\beta})J = \mathbf{0}$ we have $X \circ I = O$ and $X \circ A = O$. Partition X to conform with that of A, namely,

$$X = \left[\begin{array}{c|c|c|c|c} 0 & \boldsymbol{y}_1^T & \boldsymbol{y}_2^T & \cdots & \boldsymbol{y}_k^T \\ \hline \boldsymbol{y}_1 & X_1 & X_{12} & \cdots & X_{1k} \\ \hline \boldsymbol{y}_2 & X_{21} & X_2 & \cdots & X_{2k} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline \boldsymbol{y}_k & X_{k1} & X_{k2} & \cdots & X_k \end{array}\right], \tag{6}$$

where each $X_{ij}^T = X_{ji}$. Note that $[A, p(A)] = O$, and thus $[A, X] = [A, q(B)]$, which is of the form

$$[A,X]=\left[\begin{array}{c|c|c|c|c} 0 & \mathbf{a}_1^{\mathbf{T}}q(A_1) & \mathbf{0} & \cdots & \mathbf{0} \\ \hline q(A_1)\mathbf{a}_1 & O & O & \cdots & O \\ \hline \mathbf{0} & O & O & \cdots & O \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{0} & O & O & \cdots & O \end{array}\right]. \tag{7}$$

Note that in the $(2,2)$ block of $[A,X]=[A,q(B)]$ we have $[A_1,q(A_1)]=O$. Further note that for $\ell=1,2,\dots,k$ the $(\ell+1,\ell+1)$-block of $[A,X]$ is $[A_\ell,X_\ell]-\boldsymbol{y}_\ell\boldsymbol{a}_\ell^T+\boldsymbol{a}_\ell\boldsymbol{y}_\ell^T=O$. Thus, $[A_\ell,X_\ell]=\boldsymbol{y}_\ell\boldsymbol{a}_\ell^T-\boldsymbol{a}_\ell\boldsymbol{y}_\ell^T$. But only the first entry of $\boldsymbol{a}_\ell$ is nonzero. Thus $(\boldsymbol{y}_\ell\boldsymbol{a}_\ell^T-\boldsymbol{a}_\ell\boldsymbol{y}_\ell^T)(1)=O$.

Hence for each $\ell=1,2,\dots,k$ we have

(a) $[A_\ell,X_\ell](\ell)=O$;
(b) $X_\ell\circ I=O$;
(c) $X_\ell\circ A_\ell=O$ and
(d) A_ℓ has the Duarte property with respect to the vertex v_ℓ (by Corollary 2.5 and the fact that A has the Duarte property with respect to vertex v_1).

By [4, Lemma 2.2] we have $X_\ell=O$ for each $\ell=1,2,\dots,k$. But that means each $\boldsymbol{a}_\ell\boldsymbol{y}_\ell^T=\mathbf{0}$. Recall that each $\boldsymbol{a}_\ell$ has only one nonzero entry, which is its first entry. Consequently, $\boldsymbol{y}_\ell=\mathbf{0}$, for $\ell=1,2,\dots,k$.

Now consider the $(i+1,j+1)$-block of $[A,X]$, where $1\le i<j\le k$. This block is $A_iX_{ij}-X_{ij}A_j=O$. Since vertex v_1 is a Duarte vertex for A, $\sigma(A_i)\cap\sigma(A_j)=\emptyset$ for $2\le i<j\le k$, and hence by [4, Lemma 1.1] $X_{ij}=O$, for all $2\le i<j\le k$. Since $X_{ji}=X_{ij}^T$, we get $X_{ij}=O$ for $i,j>1$.

Furthermore, $A_1X_{1j}=X_{1j}A_j$ for $j=1,2,\dots,k$. By [4, Lemma 1.1] there are eigenvectors $\boldsymbol{u}_1,\boldsymbol{u}_2,\dots,\boldsymbol{u}_s$ of A_1 and $\boldsymbol{w}_1,\boldsymbol{w}_2,\dots,\boldsymbol{w}_s$ of A_j corresponding to the common eigenvalues of A_1 and A_j, and scalars $c_1,c_2,\dots,c_s$ such that

$$X_{1j}=\sum_{\ell=1}^{s}c_\ell\boldsymbol{u}_\ell\boldsymbol{w}_\ell^T.$$

Since A_j $(j\ge 2)$ has the Duarte property with respect to a vertex, Corollary 2.6 implies that A_j $(j\ge 2)$ has distinct eigenvalues. Hence, $\boldsymbol{w}_1,\boldsymbol{w}_2,\dots,\boldsymbol{w}_s$ are linearly independent.

Moreover, each A_ℓ has the Duarte property with respect to vertex v_ℓ which corresponds to its first entry. Thus $\boldsymbol{e}^T\boldsymbol{u}_\ell \neq 0$, for $\ell = 1, 2, \dots, k$, where $\boldsymbol{e}$ is the standard unit vector of the appropriate size with a 1 on the first entry. The $(1, j+1)$-block of $[A, X]$ is $\boldsymbol{a}_j^T X_{1j} = O$. Since $\boldsymbol{a}_j$ has only one nonzero entry on the first position, $\boldsymbol{e}^T X_{1j} = O$ for $j = 1, 2, \dots, k$. Thus

$$\boldsymbol{e}^T X_{1j} = \sum_{\ell=1}^{s} c_\ell \boldsymbol{e}^T \boldsymbol{u}_\ell \boldsymbol{w}_\ell^T = O.$$

Since $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_k\}$ is linearly independent and each $\boldsymbol{e}^T\boldsymbol{u}_\ell$ is nonzero, $c_\ell = 0$ for $\ell = 1, 2, \dots, k$. Thus, $X_{1j} = O$ for $\ell = 1, 2, \dots, k$. This shows that $X = O$.

Recall that $X = p(A) + q(B)$. Since $X = O$, $p(A) = -q(B)$, which has first row and column all zeros. By Lemma 3.1 $q(x)$ is the zero polynomial. In particular, $p(A) = -q(B) = O$. Since $\deg(p) < n$ and A has n distinct eigenvalues and $p(A) = O$, we get $p(x)$ is the zero polynomial. That is $\boldsymbol{\alpha} = \boldsymbol{0}$ and $\boldsymbol{\beta} = \boldsymbol{0}$. □

Theorem 3.3. *Let T be a tree on n vertices, and v be a fixed vertex of T with neighbors $v_1, v_2, \dots, v_k$, and let $T_i = T_{v_i}(v)$. Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be a set of n distinct real numbers. Also, let A be a real symmetric matrix whose graph is T, its spectrum is Λ, and v_1 is a Duarte vertex for A. Then there exists a perturbation $\widetilde{A}$ of A such that*

- (i) *graph of $\widetilde{A}$ is also T,*
- (ii) *eigenvalues of $\widetilde{A}$ are the same as eigenvalues of A,*
- (iii) *every vertex that is a Duarte vertex for A is a Duarte vertex for $\widetilde{A}$, and*
- (iv) *vertex v is also a Duarte vertex for $\widetilde{A}$.*

Proof. First consider the case that A_1 and A do not have any common eigenvalues. Then by Corollary 2.3, A and $A(v)$ don't have any common eigenvalues, and hence A has the Duarte property with respect to v. Thus we may take $\widetilde{A} = A$ and clearly (i)–(iv) hold.

Next consider the case that A_1 and A have at least one common eigenvalues. Let f be defined as in (5). By Lemma 3.2 the rows of $\mathrm{Jac}(f)\Big|_A$ are linearly independent, that is $\mathrm{Jac}(f)\Big|_A$ is onto. Note that

$$f\Big|_A = \left(\frac{\sum \lambda_i}{2}, \frac{\sum \lambda_i^2}{4}, \dots, \frac{\sum \lambda_i^n}{2n}, \frac{\sum \mu_i}{2}, \frac{\sum \mu_i^2}{4}, \dots, \frac{\sum \mu_i^m}{2m}\right),$$

where μ_i's are the eigenvalues of A_1. Choose $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_m\}$ such that γ_i is *close* but not equal to μ_i, for $i = 1, 2, \dots, m$, and that $\Gamma \cap \Lambda = \emptyset$, and $\Gamma \cap \sigma(A_j) = \emptyset$, for $j = 2, 3, \dots, k$. Then by the Implicit Function Theorem there is a small perturbation $\widetilde{A}$ in $\mathbb{R}^{2n-1}$ of A, such that

$$f\Big|_{\widetilde{A}} = \left(\frac{\sum \lambda_i}{2}, \frac{\sum \lambda_i^2}{4}, \dots, \frac{\sum \lambda_i^n}{2n}, \frac{\sum \gamma_i}{2}, \frac{\sum \gamma_i^2}{4}, \dots, \frac{\sum \gamma_i^m}{2m}\right).$$

That is, properties (i) and (ii) hold. Newton's identities imply that the $\sigma(\widetilde{A}_1) = \Gamma$. Since the perturbation is small, $\sigma(\widetilde{A}(w))$ remains close to $\sigma(A(w))$. That is, if w is a Duarte vertex for A, it is also a Duarte vertex for $\widetilde{A}$. Hence property (iii) holds.

Recall that by Corollary 2.6 since A has the Duarte property with respect to vertex v_1, $A_1, A_2, \ldots, A_k$ have distinct eigenvalues. If v is not a Duarte vertex for $\widetilde{A}$, then by Corollary 2.3 there is a vertex u with two connected components not containing v which have a common eigenvalue. Since $\widetilde{A}$ has the Duarte property with respect to vertex v_1, this could happen only if $u = 1$ and one of the connected components is T_1 and another is T_j, for some $j = 2, 3, \ldots, k$. But $\widetilde{A}$ is constructed in a way that $\widetilde{A}_1$ and $\widetilde{A}_j$ do not have a common eigenvalue. Hence property (iv) also holds. □

Now we are ready to solve the nowhere-zero eigenbasis problem for trees.

Theorem 3.4. *For a given tree T on n vertices $1, 2, \ldots, n$, and given distinct eigenvalue $\lambda_1, \lambda_2, \ldots, \lambda_n$, there exists a real symmetric matrix A whose graph is T and its eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n$, such that none of the eigenvectors of A has a zero entry.*

Proof. By [4, Theorem 4.2] there is a real symmetric matrix A whose graph is T and its eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n$, and A has the Duarte property with respect to vertex 1. Apply Theorem 3.3 to vertex 1 and a neighbor of it which is not a Duarte vertex for A, say vertex 2, to obtain a matrix $\widetilde{A}$ with the desired graph and spectrum. Matrix $\widetilde{A}$ has the Duarte property with respect to vertices 1 and 2. Replacing A by $\widetilde{A}$, we can iterate this process for a vertex that is a Duarte vertex and a neighbor of it which is not a Duarte vertex for the new A to obtain a new $\widetilde{A}$. In less than n iterations we stop and the result is a real symmetric matrix whose graph is T, its eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n$, and it has the Duarte property with respect to each vertex. That is, none of its eigenvalues has a zero entry. □

4. The Jacobian method and connected graphs

In this section we will use the results of the previous section and several results from [3,4] to show the existence of a nowhere-zero eigenbasis for any set of distinct eigenvalues for any connected graph.

Fix T to be a tree with vertices $1, 2, \ldots, n$ and edges $e_k = \{i_k, j_k\}$, for $k = 1, \ldots, n-1$. Also fix G to be a supergraph of T with m additional edges. Let $x_1, x_2, \ldots, x_{2n-1}, y_1, y_2, \ldots, y_m$ be independent indeterminates, and set

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_{2n-1}), \text{ and } \boldsymbol{y} = (y_1, y_2, \ldots, y_m).$$

Define $M = M(\boldsymbol{x}, \boldsymbol{y})$ to be the matrix with $2x_i$ in the (i, i) position for $i = 1, 2 \ldots, n$, x_{n+k} in the (i_k, j_k) and (j_k, i_k) positions, for $k = 1, 2, \ldots, n-1$, y_k in the (i_k, j_k) and (j_k, i_k) positions, where $\{i_k, j_k\}$ is an edge of G not in T, for $k = 1, 2, \ldots, m$, and zeros elsewhere. Set $N = N(\boldsymbol{x}, \boldsymbol{y}) = M(w)$; that is, N is the principal submatrix obtained from M by deleting its w-th row and column.

Let

$$t^n + c_{n-1}(\boldsymbol{x}, \boldsymbol{y})\, t^{n-1} + \cdots + c_1(\boldsymbol{x}, \boldsymbol{y})\, t^1 + c_0(\boldsymbol{x}, \boldsymbol{y})$$

and

$$t^{n-1} + d_{n-1}(\boldsymbol{x}, \boldsymbol{y})\, t^{n-2} + \cdots + d_1(\boldsymbol{x}, \boldsymbol{y})\, t + d_0(\boldsymbol{x}, \boldsymbol{y})$$

be the characteristic polynomials of M and N, respectively. Also, let $g : \mathbb{R}^{2n-1} \times \mathbb{R}^m \to \mathbb{R}^{2n-1}$ be the polynomial map defined by

$$g(\boldsymbol{x}, \boldsymbol{y}) = (c_0(\boldsymbol{x}, \boldsymbol{y}), \ldots, c_{n-1}(\boldsymbol{x}, \boldsymbol{y}), d_0(\boldsymbol{x}, \boldsymbol{y}), \ldots, d_{n-2}(\boldsymbol{x}, \boldsymbol{y}))\,. \tag{8}$$

Let $f : \mathbb{R}^{2n-1} \times \mathbb{R}^m \to \mathbb{R}^{2n-1}$ be the polynomial map defined by

$$f(\boldsymbol{x}, \boldsymbol{y}) = \left(\frac{\operatorname{tr} M}{2}, \frac{\operatorname{tr} M^2}{4}, \ldots, \frac{\operatorname{tr} M^n}{2n}, \frac{\operatorname{tr} N}{2}, \frac{\operatorname{tr} N^2}{4}, \ldots, \frac{\operatorname{tr} N^{n-1}}{2(n-1)} \right). \tag{9}$$

By Newton's identities [3, Proposition 1.2.2], there is an infinitely differentiable, invertible function $h : \mathbb{R}^{2n-1} \to \mathbb{R}^{2n-1}$ such that $g \circ h = f$. Thus, the Jacobian matrix of f at a point x is nonsingular if and only if the Jacobian matrix of g at $h(x)$ is nonsingular.

Theorem 4.1. *[4, Theorem 3.3] Let A be a matrix whose graph is a tree T with the Duarte property with respect to a vertex w, and $B = A(w)$. Let function f be defined by (9). Then $\operatorname{Jac}(f)\big|_A$ has full row rank.*

In [3] the Implicit Function Theorem is used to show the following result.

Theorem 4.2. *[3, Remark 3.3.2] Let A be a matrix whose graph is a tree T with the Duarte property with respect to a vertex w. Then for every supergraph G of T, there is a matrix $\bar{A}$ whose graph is G, and $\sigma(\bar{A}) = \sigma(A)$, and $\sigma(\overline{A}(w)) = \sigma(A(w))$. Furthermore, $\overline{A}$ can be taken to be arbitrarily close to A, entry-wise.*

We note for $\overline{A}$ sufficiently close to A, if a submatrix of A has distinct eigenvalues then so does the corresponding principal submatrix of $\overline{A}$, and if the eigenvalues of $A(i)$ strictly interlace those of A then the eigenvalues of $\overline{A}(i)$ strictly interlace those of $\overline{A}$.

Theorem 4.3. *For a given connected graph G on n vertices, and given distinct eigenvalue $\lambda_1, \lambda_2, \ldots, \lambda_n$, there exists a real symmetric matrix A whose graph is G and its eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n$, such that none of the eigenvectors of A has a zero entry.*

Proof. Let T be a spanning tree of G. By Theorem 3.4 there is a matrix A with spectrum Λ whose graph is T and all of the eigenvectors of A are nowhere-zero. This means that A has the Duarte property with respect to each vertex. Then by Theorem 4.1 $\mathrm{Jac}(f)\Big|_A$ has full row rank, for f defined by (9). By Remark 4.2 any supergraph G of T can be realized by a matrix $\overline{A}$ with the same spectrum as A, and the spectrum of $\overline{A(v)}$ arbitrarily close to spectrum of $A(v)$, for all v. That is, if an entry of an eigenvector of A is nonzero, it remains nonzero in the corresponding eigenvector of $\overline{A}$. Thus $\bar{A}$ is a matrix whose eigenvalues are $\lambda_1, \ldots, \lambda_n$, its graph is G, and none of its eigenvectors has a zero entry. □

5. An application

A recent paper of Ahmadi et al. [1] studies the following problem: determine, $q(G)$, the smallest number of distinct eigenvalues that a symmetric matrix with graph G has. The join of two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, denoted by $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{g, h\} | g \in V(G), h \in V(H)\}$. In particular, in [1] it is shown that for each connected graph G, $q(G \vee G) = 2$, where $G \vee G$ is the join of G with itself. We use Theorem 4.3 to extend this result. First, let us prove the following technical lemma.

Lemma 5.1. *Let $d_1, \ldots, d_n$ be distinct real numbers in the interval $(0, 1)$, $c_1, \ldots, c_n$ be real numbers, and $f(t) = \sum_{j=1}^{n} c_j \sqrt{d_j + t}$. If $f(t) = 0$ for all t in an open neighborhood of 0, then $c_1, \ldots, c_n = 0$.*

Proof. Assume that $f(t) = 0$ for all t of sufficiently small modulus. Then

$$f(0) = 0, f'(0) = 0, \ldots, f^{(n-1)}(0) = 0. \tag{10}$$

Note that for $m = 1, 2, \ldots$ we have

$$f^{(m)}(0) = \sum_{j=1}^{n} \alpha_m d^{\frac{-2m+1}{2}} c_j, \text{ where } \alpha_m = (-1)^{m-1} \frac{(2m-3)!}{2^{2m-2}(m-2)!}.$$

Equations (10) are equivalent to

$$\mathrm{diag}(\alpha_0, \ldots, \alpha_{n-1})\, V\, \mathrm{diag}(\sqrt{d_1}, \ldots, \sqrt{d_n}) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $\alpha_0 = 1$, and V is the transpose of the $n \times n$ Vandermonde matrix for $(1/d_1), \ldots, (1/d_n)$. The matrix V is invertible since d_j's are distinct. The fact that α_m's and d_j's are all nonzero and the invertibility of V imply that $c_j = 0$ for $j = 1, 2, \ldots, n$. □

Theorem 5.2. *Let G and H be connected graphs on n vertices. Then $q(G \vee H) = 2$.*

Proof. Since a matrix only has one distinct eigenvalue if and only if it is a scalar matrix, $q(G \vee H) \geq 2$.

Let $\lambda_1, \ldots, \lambda_n$ be n distinct real numbers in the interval $(0, 1)$. By Theorem 4.3 there exist symmetric matrices A and B such that A has graph G, B has graph H, both A and B have eigenvalues $\lambda_1, \ldots, \lambda_n$ and both have a nowhere zero eigenbasis. Let $A = \sum_{j=1}^{n} \lambda_j q_j q_j^T$ and $B = \sum_{j=1}^{n} \lambda_j r_j r_j^T$ be the spectral decompositions of A and B respectively.

Consider

$$M_t = \left[\begin{array}{c|c} A & \sum_{j=1}^{n} \left(\sqrt{1-\lambda_j^2+t}\right) q_j r_j^T \\ \hline \sum_{j=1}^{n} \left(\sqrt{1-\lambda_j^2+t}\right) r_j q_j^T & -B \end{array}\right].$$

Note that

$$\left[\begin{array}{c|c} \sum_{j=1}^{n} q_j q_j^T & O \\ \hline O & \sum_{j=1}^{n} r_j r_j^T \end{array}\right] M_t \left[\begin{array}{c|c} \sum_{j=1}^{n} q_j q_j^T & O \\ \hline O & \sum_{j=1}^{n} r_j r_j^T \end{array}\right]$$

is equal to

$$\left[\begin{array}{c|c} \operatorname{diag}(\lambda_1, \ldots, \lambda_n) & \operatorname{diag}\left(\sqrt{1-\lambda_1^2+t}, \ldots, \sqrt{1-\lambda_n^2+t}\right) \\ \hline \operatorname{diag}\left(\sqrt{1-\lambda_1^2+t}, \ldots, \sqrt{1-\lambda_n^2+t}\right) & \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \end{array}\right].$$

Thus, M_t is similar to

$$\bigoplus_{j=1}^{n} \begin{bmatrix} \lambda_j & \sqrt{1-\lambda_j^2+t} \\ \sqrt{1-\lambda_j^2+t} & -\lambda_j \end{bmatrix}.$$

This matrix has eigenvalues $\pm\sqrt{1+t}$. Hence $q(M_t) = 2$.

Since each of $q_1, \ldots, q_n$, $r_1, \ldots, r_n$ is nowhere zero, Lemma 5.1 implies that there is a t such that $\sum_{j=1}^{n} \left(\sqrt{1-\lambda_j^2+t}\right) q_j r_j^T$ has no entry equal to 0. For such t, the graph of M_t is $G \vee H$. Hence $q(G \vee H) = 2$. $\square$

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