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Spectral characterization of matchings in graphs

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A spectral characterization of the matching number (the size of a maximum matching) of a graph is given. More precisely, it is shown that the graphs G of order n whose matching number is k are precisely those graphs with the maximum skew rank $2k$ such that for any given set of k distinct nonzero purely imaginary numbers there is a real skew-symmetric matrix A with graph G whose spectrum consists of the given k numbers, their conjugate pairs and $n - 2k$ zeros.

1. Introduction

A *matching* in a graph G is a set of vertex-disjoint edges. A *maximum matching* in G is a matching with the maximum number of edges among all matchings in G.

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A *perfect matching* in a graph G on n vertices is a maximum matching consisting of $\frac{n}{2}$ edges. Matchings are well-studied combinatorial objects with practical applications such as Hall's marriage theorem (1935). For a full treatment of matchings see [6]. In 1947 Tutte gave necessary and sufficient conditions for a graph to have a perfect matching.

Theorem 1.1. *(See [7].) A graph G has a perfect matching if and only if for each vertex subset S of G, the number of odd components of $G - S$ is at most $|S|$.*

The matching number, denoted by $\text{match}(G)$, of a graph G is the number of edges in a maximum matching in G. So Theorem 1.1 characterizes all graphs G on n vertices with $\text{match}(G) = \frac{n}{2}$. In this article we give another set of necessary and sufficient conditions for a graph G to have a perfect matching. These conditions concern eigenvalues of skew-symmetric matrices corresponding to G. For a given positive integer k, we also give necessary and sufficient conditions for a graph G to have $\text{match}(G) = k$.

We begin by introducing some required terminology as given in [2]. Let $A = [a_{ij}]$ be an $n \times n$ real skew-symmetric matrix. The *order* of A is n, and we denote it by $|A|$. The *graph of* A, denoted by $G(A)$, has the vertex set $\{1, 2, \ldots, n\}$ and the edge set $\{\{i, j\} : a_{ij} \neq 0, 1 \leq i < j \leq n\}$. The set $S^-(G)$ denotes the set of all real skew-symmetric matrices whose graph is G. The *maximum skew rank* of G, denoted by $\text{MR}^-(G)$, is defined to be $\max\{\text{rank}(A) : A \in S^-(G)\}$. The maximum skew rank and the matching number of a graph are related as follows.

Theorem 1.2. *(See [4, Theorem 2.5].)* $\text{MR}^-(G) = 2\text{match}(G)$ *for all graphs G.*

The rank of a real symmetric or skew-symmetric matrix can be determined by its nonzero eigenvalues as follows.

Lemma 1.3. *(See [1, Corollary 2.5.14].) Let A be a real symmetric or skew-symmetric matrix. Then* $\text{rank}(A)$ *equals to the number of nonzero eigenvalues of A.*

A *full matching* in a graph G on n vertices is a matching M such that $2|M| = n$ or $n-1$, i.e., $\text{match}(G) = \lfloor \frac{n}{2} \rfloor$. In Section 2 we determine existence of a full matching of G using nonzero eigenvalues of matrices in $S^-(G)$. In Section 3, for a given positive integer k, we give necessary and sufficient conditions for G, in terms of nonzero eigenvalues of matrices in $S^-(G)$, to have $\text{match}(G) = k$.

To study matchings in connected graphs we first study matchings in trees. A certain kind of trees called NEB trees is introduced in [2] and it has been shown that any NEB tree has a full matching. We introduce required definitions and notation for NEB trees as given in [2].

Notation: Let T be a tree, and let $T(v)$ denote the forest obtained from T by deleting vertex v. Also, let $T' = T_w(v)$ denote the connected component of $T(v)$ that contains

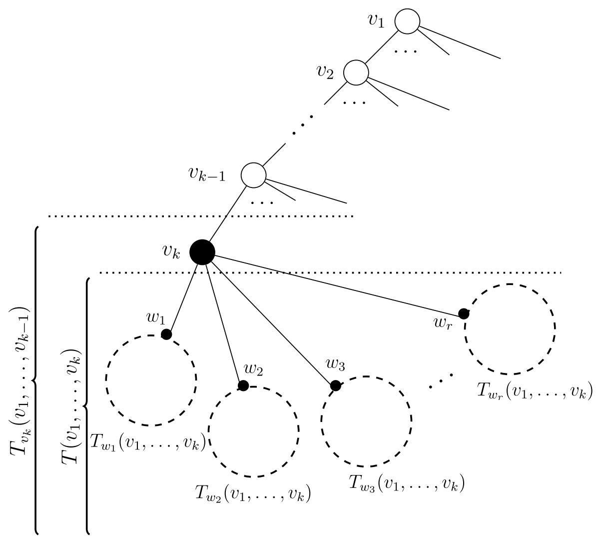


Fig. 1. Tree T with subgraphs $T(v_1,\dots,v_k)$ and $T_{v_k}(v_1,\dots,v_{k-1})$.

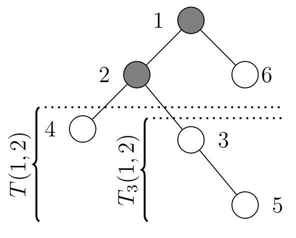


Fig. 2. Tree T with subgraphs $T(1,2)$ and $T_3(1,2)$.

the neighbor w of v. T' is a tree, hence it makes sense to consider $T'(w) = (T_w(v))\,(w)$, the forest obtained from T' by deleting vertex w, and $T'' = (T_w(v))_u\,(w)$, the connected component of $T'(w)$ that contains the neighbor u of w, and so on. For simplicity, we denote the tree $(\cdots(((T_{v_2}(v_1))_{v_3}(v_2))_{v_4}(v_3))\cdots)_{v_k}(v_{k-1})$ by $T_{v_k}(v_1,v_2,\dots,v_{k-1})$, and the forest $((\cdots(((T_{v_2}(v_1))_{v_3}(v_2))_{v_4}(v_3))\cdots)_{v_k}(v_{k-1}))(v_k)$ by $T(v_1,\dots,v_k)$. See Fig. 1.

Example 1.4. Consider the graph in Fig. 2. Delete vertex 1 and consider the connected component that contains the vertex 2. This tree is denoted by $T_2(1)$. Then in this tree delete vertex 2. The obtained forest is denoted by $T(1,2)$. The connected component of $T(1,2)$ that contains the vertex 3 is denoted by $T_3(1,2)$.

Definition 1.5. (See [2, Definition 2.3].) Let T be a tree on n vertices, and w be a vertex of T. T is defined to have *nearly even branching property at w* (in short, T is *NEB at w*) as follows. If $n = 1$, T is NEB at w. If $n \geq 2$, T is NEB at w if the following conditions are satisfied:

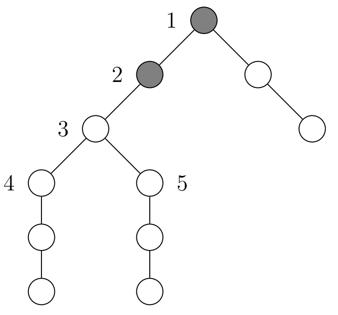


Fig. 3. Subtree $T_3(1,2)$ is a minimal non-NEB subtree of T with respect to vertex 1.

(i) $T(w)$ has exactly one odd component if n is even, and $T(w)$ has no odd component if n is odd; and
(ii) for each neighbor v of w in T, $T_v(w)$ is NEB at v.

Observation 1.6. *If a tree T is not NEB with respect to a vertex v, then there is a vertex w such that $T(w)$ has at least two odd components.*

Proof. Let $v_1 = v$. If $T(v_1)$ has at least two odd components, then $w = v_1$. Otherwise there are vertices $v_2, \dots, v_k$ such that $T(v_1, v_2, \dots, v_k)$ has at least two odd connected components. Let $w = v_k$. Now $T(w)$ has one more branch (at v_{k-1}) than $T_{v_k}(v_1, v_2, \dots, v_{k-1})$, thus it has at least two odd components. $\square$

For a vertex v, let $\mathrm{N}(v)$ denote the set of all neighbors of v. Let T be a tree which is not NEB at a vertex v_1. There exists $v_2, v_3, \dots, v_k$ such that $T_{v_k}(v_1, v_2, \dots, v_{k-1})$ is not NEB at v_k, but every $T_w(v_1, v_2, \dots, v_k)$ is NEB at w for all $w \in \mathrm{N}(v_k) \setminus \{v_{k-1}\}$. We call such $T_{v_k}(v_1, \dots, v_{k-1})$ a minimal non-NEB subtree (with respect to v_1).

Example 1.7. Tree T shown in Fig. 3 is not NEB at vertex 1 because $T_3(1,2)$ is not NEB with respect to vertex 3. But $T_4(1,2,3)$ and $T_5(1,2,3)$ both are NEB with respect to 4 and 5, respectively. Hence, $T_3(1,2)$ is a minimal non-NEB subtree of T with respect to vertex 1.

The following theorem gives the most important known result we use in this article. It shows that if a tree T is NEB at a vertex, then T has a full matching.

Theorem 1.8. *(See [2, Corollary 5.3].) Let G be a connected graph on n vertices and λ_1, λ_2, …, λ_n distinct real numbers such that*

$$\lambda_j = -\lambda_{n+1-j},$$

for all $j = 1, \dots, n$. If G has a spanning tree which is NEB at a vertex, then $\mathrm{match}(G) = \lfloor \frac{n}{2} \rfloor$ *and there exists a matrix $A \in S^-(G)$ with eigenvalues $\mathrm{i}\lambda_1, \dots, \mathrm{i}\lambda_n$.*

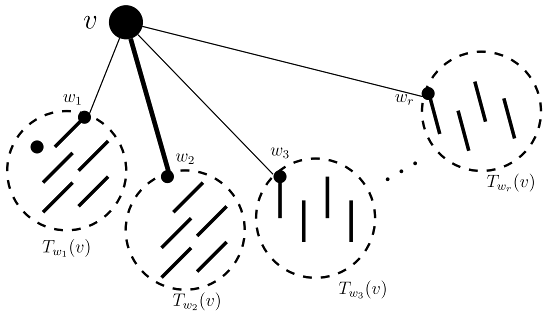


Fig. 4. Vertex w_1 is not matched with vertex v where $T_{w_1}(v)$ is an odd component.

2. Characterizations of NEB trees and connected graphs with a perfect matching

Theorem 1.8 shows that if a tree T is NEB at a vertex, then T has a full matching. It is natural to ask if the converse is true. In the next theorem we show that the converse is indeed true.

Theorem 2.1. *Let T be a tree on n vertices. Tree T is NEB with respect to some vertex v if and only if* $\operatorname{match}(T) = \lfloor \frac{n}{2} \rfloor$.

Proof. The forward direction is proved in [2, Observation 3.8]. For the backward direction, assume T is not NEB with respect to any vertex. By Observation 1.6 there is a vertex v of T such that $T(v)$ has at least two odd components. Let $T_{w_1}(v)$ and $T_{w_2}(v)$ be two such odd components.

There are two cases:

Case 1: n is even. Thus, $\lfloor \frac{n}{2} \rfloor = \frac{n}{2}$, that is, T has a perfect matching, and exactly one of the neighbors of v is matched with v. That is, at least one of the w_1 or w_2 are not matched with v. Without loss of generality, assume that w_1 is the vertex which is not matched (see Fig. 4). Then $T_{w_1}(v)$ is a tree with odd number of vertices, hence it has a vertex which is not matched. Furthermore, since T has an even number of vertices, it has at least 2 vertices which are not matched. That contradicts the assumption that T has a perfect matching.

Case 2: n is odd. Fix v_1 and find a minimal non-NEB subtree of T (with respect to v_1), say $T_{v_k}(v_1, \ldots, v_{k-1}) = T'$. Let $v = v_k$. Since T' is a minimal non-NEB subtree of T, $T'(v)$ has at least two odd components.

(a) $T'(v)$ has at least 3 odd components, then similar to Case 1, v is matched with at most one of its neighbors in an odd component, and other two odd components each have at least one vertex which is not matched. Hence $\operatorname{match}(T) < \lfloor \frac{n}{2} \rfloor$.

(b) $T'(v)$ has exactly two odd components, say $T'_{w_1}(v)$ and $T'_{w_2}(v)$. Now, consider $T_v(w_1)$ (see Fig. 5), which has even number of vertices. If $T_v(w_1)$ is

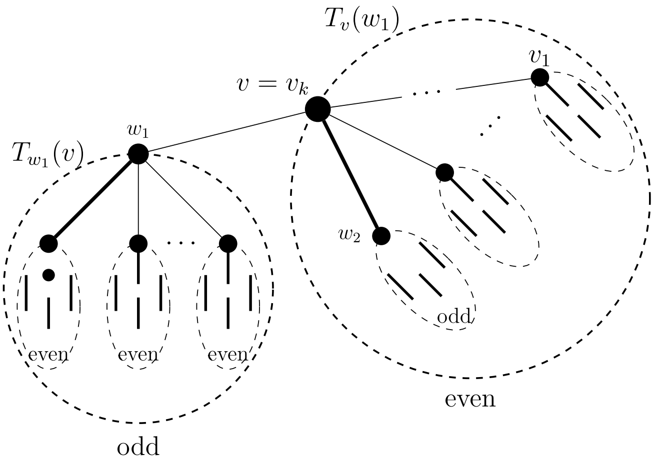


Fig. 5. Tree T and subtrees $T_v(w_1)$ and $T_{w_1}(v)$.

NEB at v, then T is NEB at w_1 by minimality of T'. Otherwise, $T_v(w_1)$ has at least two vertices which are not matched, by Case 1. Furthermore, since T has odd number of vertices, then it has at least 3 vertices which are not matched. Thus $\operatorname{match}(T) < \lfloor \frac{n}{2} \rfloor$. $\square$

We get the following corollary from Theorem 1.8 and Theorem 2.1.

Corollary 2.2. *Let T be a tree on n vertices. Then* $\operatorname{match}(T) = \lfloor \frac{n}{2} \rfloor$ *if and only if there is a real skew-symmetric matrix A with distinct eigenvalues whose graph is T.*

Below we mention a rather easy exercise in graph theory, and we will use it to extend the above result to connected graphs.

Lemma 2.3. *Let G be a connected graph on n vertices. Then* $\operatorname{match}(G) = \lfloor \frac{n}{2} \rfloor$ *if and only if G has a spanning tree T with* $\operatorname{match}(T) = \lfloor \frac{n}{2} \rfloor$*. More specifically, for any matching M (of any size) of G, there is a spanning tree T of G which includes all the edges of M.*

Proof. If a spanning tree of G has a full matching, then G has a full matching. Fix a matching M of G. Every cycle of G contains an edge which is not in M. Delete one such edge from G, and repeat this process with the obtained graph which is still connected, it contains all edges of M, and it has at least one less cycle than G. The process stops with a connected acyclic graph (tree) on n vertices, since G has finitely many cycles. The obtained graph is a spanning tree of G which contains all edges of M. $\square$

Theorem 2.4. *Let G be a connected graph on n vertices. If* $\operatorname{match}(G) = \lfloor \frac{n}{2} \rfloor$*, then for any n distinct real numbers $\lambda_1, \ldots, \lambda_n$ such that $\lambda_j = -\lambda_{n+1-j}$ for all $j = 1, \ldots, n$, there* *is a matrix* $A \in S^-(G)$ *with eigenvalues* $\mathrm{i}\lambda_1, \dots, \mathrm{i}\lambda_n$*. Conversely if there is a matrix* $A \in S^-(G)$ *with distinct eigenvalues, then* $\mathrm{match}(G) = \lfloor \frac{n}{2} \rfloor$.

Proof. Assume that $\mathrm{match}(G) = \lfloor \frac{n}{2} \rfloor$. By Lemma 2.3 graph G has full matching if and only if it has a spanning tree T with a full matching. Also by Theorem 2.1, T has a full matching if and only if T is NEB with respect to a vertex. Thus, by Theorem 1.8, G realizes a real skew-symmetric matrix A with the given eigenvalues.

Conversely suppose that there is a real skew-symmetric matrix A with distinct eigenvalues whose graph is G. Then, by Lemma 1.3 and Theorem 1.2,

$$2\left\lfloor \frac{n}{2} \right\rfloor = \mathrm{rank}(A) \leq \mathrm{MR}^-(G) = 2\mathrm{match}(G).$$

That is, $\left\lfloor \frac{n}{2} \right\rfloor \leq \mathrm{match}(G)$. Since $\mathrm{match}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$ for any graph G, we have $\mathrm{match}(G) = \left\lfloor \frac{n}{2} \right\rfloor$. $\square$

Theorem 2.4 immediately implies the following corollary giving a spectral condition for a connected graph to have a perfect matching or a near perfect matching.

Corollary 2.5. *Let G be a connected graph. Then G has a full matching if and only if there is a matrix $A \in S^-(G)$ with distinct eigenvalues.*

3. Spectral characterization of graphs with arbitrary matching number

It is known that $\mathrm{match}(G) = k$ if and only if $\mathrm{MR}^-(G) = 2k$, i.e., G realizes a skew-symmetric with $2k$ nonzero eigenvalues by Theorem 1.2 and Lemma 1.3. In this section we prove that these eigenvalues can be any k distinct nonzero purely imaginary numbers and their conjugate pairs. Similar to approaches in [2,3] we are going to use the Jacobian method, so we need to define an appropriate function and show its Jacobian is nonsingular when it is evaluated at some point.

Let G be a graph on n vertices with matching number k, and $k + m$ edges where $m > 0$. Fix a maximum matching $\mathcal{M}$ of G and without loss of generality assume $\mathcal{M} = \{\{1, 2\}, \{3, 4\}, \dots, \{2k - 1, 2k\}\}$. Assume the m edges of G that are not in $\mathcal{M}$ are of the forms $e_l = \{i_l, j_l\}$, for $l = 1, 2, \dots, m$. Let $x_1, \dots, x_k, y_1, \dots, y_m$ be $k + m$ independent indeterminates and set

$$\boldsymbol{x} = (x_1, x_2, \dots, x_k), \text{ and } \boldsymbol{y} = (y_1, y_2, \dots, y_m).$$

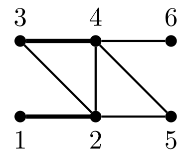
We define a skew-symmetric matrix of variables where x_j are in the positions corresponding to the edges in $\mathcal{M}$, and y_l are in the positions of the edges not in $\mathcal{M}$. Let $M = M(\boldsymbol{x}, \boldsymbol{y})$ be an $n \times n$ skew-symmetric matrix whose $(2j - 1, 2j)$-entry is x_j, $(2j, 2j - 1)$-entry is $-x_j$, for $j = 1, 2, \dots, k$, and for $l = 1, 2, \dots, m$ let the (i_l, j_l)-entry of M to be y_l where

$i_l < j_l$, and $-y_l$, otherwise. Note that Since $\mathrm{match}(G) = k$, $G - \{1, 2, \ldots, 2k\}$ has no edges. Thus M has the following form.

$$M = \left[\begin{array}{c|c} N & L \\ \hline -L^T & O \end{array}\right],$$

where N is the upper left $2k \times 2k$ block of M, O is the square zero matrix of size $n - 2k$, and L contains only y_l's and zeros. Note that N contains zero entries, all of the x_j's, and some or none of y_l's. In particular, the $(2j - 1, 2j)$-th entry of N is x_j, for $j = 1, 2, \ldots, k$.

Example 3.1. Consider the following graph G on 6 vertices with 6 edges and $\mathrm{match}(G) = 2$.



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For the above G, $\mathcal{M} = \{\{1, 2\}, \{3, 4\}\}$ is a maximum matching. So $M = M(\boldsymbol{x}, \boldsymbol{y})$ would have the following form.

$$M = M(\boldsymbol{x}, \boldsymbol{y}) = \left[\begin{array}{cccc|cc} 0 & \boldsymbol{x_1} & 0 & 0 & 0 & 0 \\ -\boldsymbol{x_1} & 0 & y_1 & y_2 & y_3 & 0 \\ 0 & -y_1 & 0 & \boldsymbol{x_2} & 0 & 0 \\ 0 & -y_2 & -\boldsymbol{x_2} & 0 & y_4 & y_5 \\ \hline 0 & -y_3 & 0 & -y_4 & 0 & 0 \\ 0 & 0 & 0 & -y_5 & 0 & 0 \end{array}\right].$$

A real evaluation A of M is obtained by assigning real values to indeterminates in $\boldsymbol{x}$ and $\boldsymbol{y}$. Clearly such evaluation A is a skew-symmetric matrix whose graph is a subgraph of G and the eigenvalue of A are purely imaginary occurring in conjugate pairs and some zeros. Define the following ordering of the purely imaginary axis of the complex plane: for two numbers a and b on the imaginary axis of the complex plane let $a \geq b$ if $-a\mathrm{i} \geq -b\mathrm{i}$ and the equality holds if and only if $a = b$.

Define $F : \mathbb{R}^{k+m} \to \mathbb{R}^n$ by

$$F(\boldsymbol{x}, \boldsymbol{y}) = \big(-\mathrm{i}\lambda_1(M), -\mathrm{i}\lambda_2(M), \ldots, -\mathrm{i}\lambda_n(M) \big),$$

where $\lambda_j(M)$ is the j-th largest eigenvalue of M. Note that, some of the middle components of F might be zero. Furthermore, since $\lambda_j(M) = -\lambda_{n-j+1}(M)$ for $j = 1, \ldots, n$, F is completely defined by half of its components, say the ones in upper half-plane and zeros. Moreover, M has at most k nonzero eigenvalues in the upper half-plane since $\mathrm{MR}^-(G) = 2\mathrm{match}(G) = 2k$. That is, F is completely determined by its first k components.

Define $f : \mathbb{R}^{k+m} \to \mathbb{R}^k$ by

$$f(\boldsymbol{x}, \boldsymbol{y}) = \big(- \mathrm{i}\lambda_1(M), -\mathrm{i}\lambda_2(M), \ldots, -\mathrm{i}\lambda_k(M)\big).$$

Let $\lambda_1 > \lambda_2 > \ldots > \lambda_k > 0$ be k distinct nonzero purely imaginary numbers. Set $\boldsymbol{a} = (-\mathrm{i}\,\lambda_1, -\mathrm{i}\,\lambda_2, \ldots, -\mathrm{i}\,\lambda_k) \in \mathbb{R}^k$, $\boldsymbol{b} = (0, \ldots, 0) \in \mathbb{R}^m$ and $A = M(\boldsymbol{a}, \boldsymbol{b})$. Then A is the block diagonal matrix

$$A = \bigoplus_{j=1}^{k} \begin{bmatrix} 0 & -\mathrm{i}\lambda_j \\ \mathrm{i}\lambda_j & 0 \end{bmatrix} \oplus O_{n-2k}. \tag{3.1}$$

That is,

$$A = \left[\begin{array}{cc|cc|c|cc|c} 0 & -\mathrm{i}\lambda_1 & 0 & 0 & \cdots & 0 & 0 & \\ \mathrm{i}\lambda_1 & 0 & 0 & 0 & \cdots & 0 & 0 & \\ \cline{1-7} 0 & 0 & 0 & -\mathrm{i}\lambda_2 & \cdots & 0 & 0 & \\ 0 & 0 & \mathrm{i}\lambda_2 & 0 & \cdots & 0 & 0 & \\ \cline{1-7} & & & & \ddots & & & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ & & & & \ddots & & & \\ \cline{1-7} 0 & 0 & 0 & 0 & \cdots & 0 & -\mathrm{i}\lambda_k & \\ 0 & 0 & 0 & 0 & \cdots & \mathrm{i}\lambda_k & 0 & \\ \hline & & & & O & & & O \end{array}\right].$$

It easy to check that the nonzero eigenvalues of A are $\pm\lambda_1, \pm\lambda_2, \ldots, \pm\lambda_k$ and consequently $f\Big|_A = f(\boldsymbol{a}, \boldsymbol{b}) = (-\mathrm{i}\,\lambda_1, -\mathrm{i}\,\lambda_2, \ldots, -\mathrm{i}\,\lambda_k)$. We want to show that the Jacobian of f evaluated at the point $(\boldsymbol{a}, \boldsymbol{b})$ is nonsingular. It is known that the eigenvalues and eigenvectors of a matrix with distinct eigenvalues are continuous differentiable functions of the entries of the matrix [8]. The following lemma shows the derivative of the nonzero eigenvalues of a skew-symmetric matrix with $2k$ distinct nonzero eigenvalues and $n - 2k$ zero eigenvalues with respect to the entries of the matrix, in terms of the entries of their corresponding eigenvectors.

Lemma 3.2. *Let A be an $n \times n$ real skew-symmetric matrix with distinct nonzero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ in the upper half-plane, and corresponding unit eigenvectors*

$\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$. Let $A(t) = A + tE_{rs} - tE_{sr}$, for $t \in (-\varepsilon, \varepsilon)$, where ε is a small positive number. Also, let $\lambda_j(t)$ be the j-th largest eigenvalue of $A(t)$ with corresponding eigenvector $\boldsymbol{v}_j(t)$, and $\boldsymbol{v}_{j_r}$ denote the r-th entry of the vector $\boldsymbol{v}_j$. Then

$$\left.\frac{d\lambda_j(t)}{dt}\right|_{t=0} = 2\mathrm{i}\,\mathrm{Im}(\overline{\boldsymbol{v}_{j_r}}\boldsymbol{v}_{j_s}),$$

where $\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.

Proof. Note that $A(t)$, $\lambda_j(t)$ and $\boldsymbol{v}_j(t)$ are continuous functions of t, so $A(0) = A$, $\lambda_j(0) = \lambda_j, \boldsymbol{v}_j(0) = \boldsymbol{v}_j$, and when $t \to 0$ we have

$$\begin{aligned} A(t) &\to A, \\ \lambda_j(t) &\to \lambda_j, \\ \boldsymbol{v}_j(t) &\to \boldsymbol{v}_j. \end{aligned}$$

Furthermore,

$$\dot{A}(0) = E_{rs} - E_{sr},$$

and

$$A(t)\boldsymbol{v}_j(t) = \lambda_j(t)\boldsymbol{v}_j(t).$$

Differentiating both sides with respect to t we get

$$\dot{A}(t)\boldsymbol{v}_j(t) + A(t)\dot{\boldsymbol{v}}_j(t) = \dot{\lambda}_j(t)\boldsymbol{v}_j(t) + \lambda_j(t)\dot{\boldsymbol{v}}_j(t).$$

Set $t = 0$, then

$$(E_{rs} - E_{sr})\boldsymbol{v}_j + A\dot{\boldsymbol{v}}_j(0) = \dot{\lambda}_j(0)\boldsymbol{v}_j + \lambda_j\dot{\boldsymbol{v}}_j(0).$$

Multiplying both sides by $\overline{\boldsymbol{v}_j}^T$ from left we get

$$\overline{\boldsymbol{v}_j}^T(E_{rs} - E_{sr})\boldsymbol{v}_j + \overline{\boldsymbol{v}_j}^T A\dot{\boldsymbol{v}}_j(0) = \dot{\lambda}_j(0)\overline{\boldsymbol{v}_j}^T\boldsymbol{v}_j + \lambda_j\overline{\boldsymbol{v}_j}^T\dot{\boldsymbol{v}}_j(0).$$

Since A is skew-symmetric $A\overline{\boldsymbol{v}_j} = -\lambda_j\overline{\boldsymbol{v}_j}$. Hence

$$\overline{\boldsymbol{v}_j}^T A = (A^T\overline{\boldsymbol{v}_j})^T = (-A\overline{\boldsymbol{v}_j})^T = (-(-\lambda_j\overline{\boldsymbol{v}_j}))^T = \lambda_j\overline{\boldsymbol{v}_j}^T.$$

Also,

$$\overline{\boldsymbol{v}_j}^T(E_{rs} - E_{sr})\boldsymbol{v}_j = \overline{\boldsymbol{v}_{j_r}}v_{j_s} - \overline{\boldsymbol{v}_{j_s}}v_{j_r} = 2\mathrm{i}\,\mathrm{Im}(\overline{\boldsymbol{v}_{j_r}}\boldsymbol{v}_{j_s}).$$

Thus

$$2\mathrm{i}\,\mathrm{Im}(\overline{\boldsymbol{v}_{j_r}}\boldsymbol{v}_{j_s}) + \lambda_j \overline{\boldsymbol{v}_j}^T \dot{\boldsymbol{v}}_j(0) = \dot{\lambda}_j(0)\overline{\boldsymbol{v}_j}^T \boldsymbol{v}_j + \lambda_j \overline{\boldsymbol{v}_j}^T \dot{\boldsymbol{v}}_j(0).$$

The second term in left hand side is equal to the second term in right hand side, and $\boldsymbol{v}_j$'s are unit vectors, that is, $\overline{\boldsymbol{v}_j}^T \boldsymbol{v}_j = 1$. Hence

$$2\mathrm{i}\,\mathrm{Im}(\overline{\boldsymbol{v}_{j_r}}\boldsymbol{v}_{j_s}) = \dot{\lambda}_j(0). \quad \square$$

Corollary 3.3. *For M, A, and λ_j's defined as above, let $r = 2l - 1$, $s = 2l$, and x_l be the entry in the (r, s) position of M. Then we have*

$$\frac{\partial}{\partial x_l}\big(-\mathrm{i}\,\lambda_j(M)\big)\Big|_A = \begin{cases} 1, & \text{if } j = l, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Note that for A we have

$$\boldsymbol{v}_j = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \cdots & 0 & \mathrm{i} & -1 & 0 & \cdots & 0 \end{bmatrix}^T,$$

where the nonzero entries are at $2j - 1$ and $2j$ positions. Also note that

$$\frac{\partial}{\partial x_l}\big(\lambda_j(M)\big)\Big|_A = \frac{\mathrm{d}\lambda_j(t)}{\mathrm{d}t}\Big|_{t=0}.$$

Then by Lemma 3.2

$$\begin{aligned}\frac{\partial}{\partial x_l}\big(-\mathrm{i}\,\lambda_j(M)\big)\Big|_A &= (-\mathrm{i})2\mathrm{i}\,\mathrm{Im}\left(\overline{\boldsymbol{v}_{j_{2l-1}}}\boldsymbol{v}_{j_{2l}}\right) \\ &= \begin{cases} 2\,\mathrm{Im}\left(\frac{-\mathrm{i}}{\sqrt{2}}\frac{-1}{\sqrt{2}}\right), & \text{if } j = l, \\ 0, & \text{otherwise.} \end{cases} \\ &= \begin{cases} 1, & \text{if } j = l, \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

This completes the proof. $\square$

Corollary 3.4. *For the matrix A and function f defined as above we have*

$$\mathrm{Jac}(f)\Big|_A = I_k,$$

where I_k denotes the $k \times k$ identity matrix. Hence, $\mathrm{Jac}(f)\Big|_A$ is nonsingular.

Now we are ready to prove the main result of this section which characterizes the graphs with matching number k. We will use the Implicit Function Theorem, mentioned below. For a full treatment of the topic see [5].

Theorem 3.5 *(Implicit Function Theorem). Let $F : \mathbb{R}^{s+r} \to \mathbb{R}^s$ be a continuously differentiable function on an open subset U of $\mathbb{R}^{s+r}$ defined by*

$$F(\boldsymbol{x}, \boldsymbol{y}) = (F_1(\boldsymbol{x}, \boldsymbol{y}), F_2(\boldsymbol{x}, \boldsymbol{y}), \dots, F_s(\boldsymbol{x}, \boldsymbol{y})),$$

where $\boldsymbol{x} = (x_1, \dots, x_s) \in \mathbb{R}^s$, $\boldsymbol{y} = (y_1, \dots, y_r) \in \mathbb{R}^r$, and F_i's are real valued multivariate functions. Let $(\boldsymbol{a}, \boldsymbol{b})$ be an element of U with $\boldsymbol{a} \in \mathbb{R}^s$ and $\boldsymbol{b} \in \mathbb{R}^r$, and $\boldsymbol{c}$ be an element of $\mathbb{R}^s$ such that $F(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{c}$. If

$$\mathrm{Jac}_x(F)\Big|_{(\boldsymbol{a},\boldsymbol{b})} = \left[\frac{\partial F_i}{\partial x_j}\Big|_{(\boldsymbol{a},\boldsymbol{b})}\right]_{s\times s}$$

is nonsingular, then there exist an open neighborhood V of $\boldsymbol{a}$ and an open neighborhood W of $\boldsymbol{b}$ such that $V \times W \subseteq U$ such that for each $\boldsymbol{y} \in W$ there is an $\boldsymbol{x} \in V$ with $F(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{c}$. Furthermore, for any $(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}) \in V \times W$ such that $F(\bar{\boldsymbol{a}}, \bar{\boldsymbol{b}}) = \boldsymbol{c}$, $\mathrm{Jac}(F)\Big|_{(\bar{\boldsymbol{a}},\bar{\boldsymbol{b}})}$ is also nonsingular.

Theorem 3.6. *Let G be a graph on n vertices, and $\lambda_1 > \lambda_2 > \ldots > \lambda_k > 0$ be k distinct nonzero purely imaginary numbers where $2k \le n$. Then $\mathrm{match}(G) = k$ if and only if*

(a) there is a matrix $A \in S^-(G)$ whose eigenvalues are $\pm\lambda_1, \pm\lambda_2, \dots, \pm\lambda_k$ and $n - 2k$ zeros, and
(b) for all matrices $A \in S^-(G)$, A has at most $2k$ nonzero eigenvalues.

Proof. Assume that (a) and (b) hold. Then (a) and Lemma 1.3 imply that $\mathrm{MR}^-(G) \ge 2k$. Furthermore (b) and Lemma 1.3 imply that $\mathrm{MR}^-(G) \le \mathrm{rank} A = 2k$. Thus $\mathrm{MR}^-(G) = 2k$. By Theorem 1.2 we have $\mathrm{match}(G) = \frac{\mathrm{MR}^-(G)}{2} = \frac{2k}{2} = k$.

Now assume that $\mathrm{match}(G) = k$. If G is a disjoint union of edges, then the matrix A given by (3.1) has the desired properties. Assume that there is an edge which is not in a maximum matching, that is, G has $k + m$ edges where $m > 0$. Consider the function f, and the matrices M and A as above. Note that $f\Big|_A = (-\mathrm{i}\lambda_1, -\mathrm{i}\lambda_2, \dots, -\mathrm{i}\lambda_k)$, and $\mathrm{Jac}(f)\Big|_A$ is nonsingular, by Corollary 3.4. Then by the Implicit Function Theorem (Theorem 3.5) there are open sets $U \in \mathbb{R}^k$ and $V \in \mathbb{R}^m$, such that $(-\mathrm{i}\lambda_1, \dots, -\mathrm{i}\lambda_k) \in U$ and $(0, \dots, 0) \in V$, and for any $(\varepsilon_1, \dots, \varepsilon_m) \in V$, there is a $(-\mathrm{i}\widehat{\lambda_1}, \dots, -\mathrm{i}\widehat{\lambda_k}) \in U$ close to $(-\mathrm{i}\lambda_1, \dots, -\mathrm{i}\lambda_k)$, such that

$$f(-\mathrm{i}\widehat{\lambda_1}, \dots, -\mathrm{i}\widehat{\lambda_k}, \varepsilon_1, \dots, \varepsilon_m) = (-\mathrm{i}\lambda_1, \dots, -\mathrm{i}\lambda_k).$$

Since V is an open neighborhood of $(0,\dots,0) \in \mathbb{R}^m$, one can choose all $\varepsilon_i \neq 0$. Let $\widehat{A} = M(-\mathrm{i}\widehat{\lambda_1},\dots,-\mathrm{i}\widehat{\lambda_k},\varepsilon_1,\dots,\varepsilon_m)$. Then eigenvalues of A are $(-\mathrm{i}\lambda_1,-\mathrm{i}\lambda_2,\dots,-\mathrm{i}\lambda_k)$ and graph of A is G. That is (a) holds. Also, by Theorem 1.2 and Lemma 1.3, (b) holds. □

Note that for a given graph G with matching number k, there might exist skew-symmetric matrices with less than $2k$ nonzero eigenvalues whose graph is G. One easy example is the complete bipartite graph $K_{n,n}, n \geq 2$. The matching number of $K_{n,n}$ is n and its skew-adjacency matrix $A = xy^T - yx^T$, where $x = \left[\, \mathbf{1} \,\middle|\, \mathbf{1} \,\right]^T$ and $y = \left[\, \mathbf{1} \,\middle|\, 2 \cdot \mathbf{1} \,\right]^T$ and $\mathbf{1}$ is the all ones vector of order n, has only two nonzero eigenvalues $\pm n\mathrm{i}$.

Remark 3.7. Theorem 3.6 shows that the graphs G of order n whose matching number is k are precisely those graphs with the maximum skew rank $2k$ such that for any given set of k distinct nonzero purely imaginary numbers there is a real skew-symmetric matrix A with graph G whose spectrum consists of the given k numbers, their conjugate pairs and $n - 2k$ zeros.

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