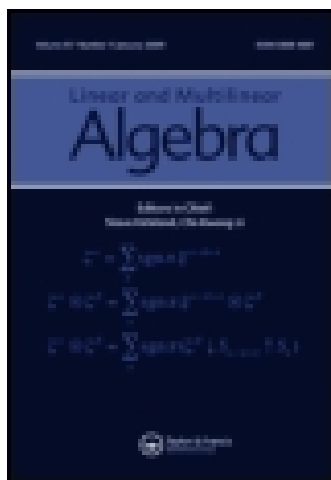


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The λ - τ structured inverse eigenvalue problem

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The $\lambda - \tau$ structured inverse eigenvalue problem

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Let $\lambda_1 < \dots < \lambda_n$ and $\tau_1 < \dots < \tau_{n-2}$ be $2n - 2$ real numbers that satisfy the strict second-order Cauchy interlacing inequalities $\lambda_i < \tau_i < \lambda_{i+2}$ for $i = 1, 2, \dots, n-2$ and the nondegeneracy conditions $\lambda_{i+1} \neq \tau_i$ for $i = 1, 2, \dots, n-2$. Given a connected graph G on n vertices with adjacent vertices i and j , it is proven that there is a real symmetric matrix A whose graph is G such that A has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and $A(\{i, j\})$ has eigenvalues $\tau_1, \tau_2, \dots, \tau_{n-2}$, provided some necessary combinatorial conditions on G are satisfied. We also provide generalizations when the two deleted vertices are not adjacent, as well as interpretation of the results in terms of perturbing one or two diagonal entries.

Keywords: graph; symmetric matrix; structured inverse eigenvalue problem; the Duarte property; the Jacobian method; Cauchy interlacing inequalities

AMS Subject Classifications: 05C50; 65F18

1. Introduction

Structured inverse eigenvalue problems are of tremendous interest and arise in various areas of research. The type of problem addressed here arises in the study of the dynamics of a system of masses and springs.[1–5] In its simplest form, the mathematical model seeks a real symmetric $n \times n$ matrix whose eigenvalues are prescribed and the eigenvalues of a given $(n - 1) \times (n - 1)$ principal submatrix are also prescribed. Physically, the problem asks if it is possible to design a spring-mass system such that the system has prescribed fundamental frequencies (that is, the corresponding matrix A has prescribed eigenvalues), and the subsystem obtained by removing one spring and one mass has prescribed fundamental frequencies (that is, the trailing $(n - 1) \times (n - 1)$ principal submatrix of A has prescribed eigenvalues).

Throughout this paper, we denote specific submatrices as follows. Let A be an $m \times n$ matrix. Assume $\alpha \subseteq \{1, 2, \dots, m\}$ and $\beta \subseteq \{1, 2, \dots, n\}$. Then

$A[\alpha; \beta]$ is the matrix obtained from A by keeping the rows indexed by α and the columns indexed by β ; and

$A(\alpha; \beta)$ is the matrix obtained from A by deleting the rows indexed by α and the columns indexed by β .

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We abbreviate $A[\alpha; \alpha]$ to $A[\alpha]$ and $A(\alpha; \alpha)$ to $A(\alpha)$. In case α or β is a singleton set, we omit the curly brackets. For example, we write $A(1)$ for $A(\{1\})$. Also in the case that α or β is empty, we may not write them. For example, $A(; 1)$ is the submatrix obtained from A by removing the first column. In case $\alpha = \beta$, we use the same notation for a graph G , where indices denote vertices. For example, $G[X]$ denotes the subgraph of G induced on the vertex set X . For other notation and definitions we follow.[6]

Let $A = [a_{ij}]$ be an $n \times n$ real symmetric matrix. The *graph* $G(A)$ of A is the graph on vertices $1, 2, \dots, n$ with i adjacent to j if and only if $a_{ij} \neq 0$ and $i \neq j$. Note that the graph does not depend on the diagonal entries of A .

The λ - μ *problem* asks if given a graph G of order n , real numbers $\lambda_1 \leq \dots \leq \lambda_n$, real numbers $\mu_1 \leq \dots \leq \mu_{n-1}$ and $i \in \{1, 2, \dots, n\}$ does there exist a real symmetric matrix A whose graph is G such that A has eigenvalues $\lambda_1, \dots, \lambda_n$ and $A(i)$ has eigenvalues μ_1, \dots, μ_{n-1} ? In 1989, Duarte solved the problem for any matrix whose graph is a tree under the necessary assumption that the μ 's interlace the λ 's, that is,

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \dots \leq \mu_{n-1} \leq \lambda_n, \quad (1)$$

and the additional assumption that the μ 's are distinct from λ 's.[7] The authors of this paper have recently extended that result to any connected graph with no multiple eigenvalues.[6] The inequalities in (1) are called the *first order Cauchy interlacing inequalities*.

One way to generalize this inverse eigenvalue problem is to ask if there is an analogue for an $n \times n$ matrix and one of its $(n-2) \times (n-2)$ submatrices. More precisely, the λ - τ *problem* asks if given a graph G of order n , real numbers $\lambda_1 \leq \dots \leq \lambda_n$, real numbers $\tau_1 \leq \dots \leq \tau_{n-2}$ and distinct i and j in $\{1, 2, \dots, n\}$, does there exist a real symmetric matrix A whose graph is G such that A has eigenvalues $\lambda_1, \dots, \lambda_n$ and $A(\{i, j\})$ has eigenvalues $\tau_1, \dots, \tau_{n-2}$?

Throughout this paper, we assume that the λ 's and the τ 's are distinct and that no λ_i and τ_j are equal. Under this assumption, we define the λ - τ *sequence* to be $X = x_1, x_2, \dots, x_{2n-2}$, where $x_1 < x_2 < \dots < x_{2n-2}$ and $\{x_1, x_2, \dots, x_{2n-2}\} = \{\lambda_1, \lambda_2, \dots, \lambda_n\} \cup \{\tau_1, \tau_2, \dots, \tau_{n-2}\}$.

In Section 2, we introduce some necessary conditions for the λ - τ problem to have a solution when the graph is a tree. In Section 3, we show that the λ - τ problem for adjacent vertices i and j and G being a tree has a solution whenever the λ 's and μ 's are distinct and certain necessary conditions are met. This is done by reducing the λ - τ problem to two λ - μ problems.[6] In Section 4, we use the Jacobian method, also used in [6], to extend the result to connected graphs. In Section 5, we extend the results of the previous two sections to the case when the vertices i and j are not adjacent. Finally, in Section 6, we use the old and new results to answer a question regarding the eigenvalues of matrix A and \hat{A} , where \hat{A} is obtained from A by perturbing one or two diagonal entries.

2. Properties of the λ - τ sequence

In this section, we derive several properties of the λ - τ sequence of an $n \times n$ symmetric matrix $A = [a_{kl}]$ and a principal submatrix $A(\{i, j\})$. Throughout this section, $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ denote the eigenvalues of A and $\tau_1 \leq \tau_2 \leq \dots \leq \tau_{n-2}$ denote the eigenvalues of $A(\{i, j\})$.

2.1. Restrictions on the λ - τ sequence

The Cauchy interlacing inequalities [8] describe some restrictions on the eigenvalues of A and $A(\{i, j\})$.

PROPOSITION 2.1 *Let A be an $n \times n$ real symmetric matrix and $A(\{i, j\})$ be a principal submatrix of A of order $n - 2$. Then the eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ of A and the eigenvalues $\tau_1 \leq \tau_2 \leq \dots \leq \tau_{n-2}$ of $A(\{i, j\})$ satisfy*

$$\lambda_i \leq \tau_i \leq \lambda_{i+2}. \tag{2}$$

We refer to the inequalities in (2) as the *second-order Cauchy interlacing inequalities*. We say $(A, A(\{i, j\}))$ is a *nondegenerate pair* if all inequalities in (2) are strict and no λ_k and τ_l are equal. Let X be the λ - τ sequence of $(A, A(\{i, j\}))$. If $(A, A(\{i, j\}))$ is a nondegenerate pair, we say X is a *nondegenerate sequence*. The inequalities (2) imply some properties about the λ - τ sequence of $(A, A(\{i, j\}))$. First we have the following.

LEMMA 2.2 *Let $X = x_1, x_2, \dots, x_{2n-2}$ be the λ - τ sequence of $(A, A(\{i, j\}))$. Assume that X is nondegenerate. Then no three consecutive x_i 's are eigenvalues of A and no three consecutive x_i 's are eigenvalues of $A(\{i, j\})$.*

Proof Consider λ_k, λ_{k+1} and λ_{k+2} . By (2), $\lambda_k \leq \tau_k \leq \lambda_{k+2}$. Thus λ_k, λ_{k+1} and λ_{k+2} do not occur consecutively in the λ - τ sequence. The case of the eigenvalues of $A(\{i, j\})$ is similar. □

If τ_k and τ_{k+1} occur consecutively in the λ - τ sequence X , then we say that (τ_k, τ_{k+1}) is a τ -pairing. If λ_k and λ_{k+1} occur consecutively in the λ - τ sequence X , then we say that $(\lambda_k, \lambda_{k+1})$ is a λ -pairing. Note by (2) that if $(\tau_k, \tau_{k+1}) = (x_l, x_{l+1})$ is a τ -pairing, then $x_{l-1} = \lambda_{k+1}$ and $x_{l+2} = \lambda_{k+2}$. Also if $(\lambda_k, \lambda_{k+1}) = (x_l, x_{l+1})$ is a λ -pairing, then $x_{l-1} = \tau_{k-1}$ and $x_{l+2} = \tau_k$.

LEMMA 2.3 *Let X be the λ - τ sequence of $(A, A(\{i, j\}))$ and assume that X is nondegenerate. The first (that is, the one with smallest x_i 's) and the last pairings of X are λ -pairings.*

Proof Suppose $(\tau_k, \tau_{k+1}) = (x_l, x_{l+1})$ is a τ -pairing of X . Then, $x_{l-1} = \lambda_{k+1}$ and thus $\{x_1, x_2, \dots, x_{l-1}\} = \{\lambda_1, \lambda_2, \dots, \lambda_{k+1}\} \cup \{\tau_1, \tau_2, \dots, \tau_{k-1}\}$ contains two more λ 's than τ 's. Hence, there is a λ -pairing in $\{x_1, x_2, \dots, x_{l-1}\}$, that is, one that precedes (τ_k, τ_{k+1}) . Similarly, there is a λ -pairing in X that follows (τ_k, τ_{k+1}) . □

Additionally, we have the following.

LEMMA 2.4 *Let X be a nondegenerate λ - τ sequence. For any two τ -pairings in X there is a λ -pairing between them and for any two λ -pairings in X there is a τ -pairing between them.*

Proof Consider two consecutive τ -pairings in X , $(\tau_k, \tau_{k+1}) = (x_r, x_{r+1})$ and $(\tau_l, \tau_{l+1}) = (x_s, x_{s+1})$. Then

$$\{x_{r+2}, x_{r+3}, \dots, x_{x_{s-1}}\} = \{\lambda_{k+2}, \lambda_{k+3}, \dots, \lambda_{l+1}\} \cup \{\tau_{k+2}, \tau_{k+3}, \dots, \tau_{l-1}\}.$$

Thus, $\{x_{r+2}, x_{r+3}, \dots, x_{x_{s-1}}\}$ has two more λ 's than τ 's and hence contains a λ -pairing. Similarly there is a τ -pairing between any two λ -pairings. \square

Lemmas 2.2–2.4 give restrictions on the choice of eigenvalues of A and $A(\{i, j\})$. A simple way to summarize the lemmas is that the τ -pairings of A and $A(\{i, j\})$ interlace the λ -pairings. Next, we use this nice property of the pairings to partition X into two sets of desired sizes such that each set includes (strictly) interlacing λ 's and τ 's. The two sets will later be used to reduce the λ - τ problem to two λ - μ problems.

LEMMA 2.5 *Let X be a nondegenerate λ - τ sequence with exactly k τ -pairings and let r and s be positive integers such that $r + s = n$ and $r, s \geq k + 1$. Then, X can be partitioned into two sets such that the first set has r λ 's and $r - 1$ τ 's and the second set s λ 's and $s - 1$ τ 's. Furthermore, in each set the τ 's and the λ 's satisfy the first order Cauchy interlacing inequalities (1).*

Proof We give an algorithm for constructing such a partition (B, C) . Since there are k τ -pairings, by Lemmas 2.3 and 2.4 there are $k + 1$ λ -pairings. First, arbitrarily assign one of the elements in each pairing to B and the other one to C . This, by Lemmas 2.3 and 2.4, results in two sets each with $k + 1$ λ 's that are interlaced by k τ 's. Let $X' = \{x'_1, x'_2, \dots, x'_{2(n-k-1)}\} = X \setminus (B \cup C)$. Thus, as long as we assign both of x'_{2l-1} and x'_{2l} to B or both to C , the τ 's in B (respectively C) will interlace the λ 's in B (respectively C). Hence, by assigning $r - k - 1$ of these pairs to B and the remaining $s - k - 1$ to C , we obtain a partition with the desired properties.

Consider two consecutive pairings in X . The portion of X between the two pairing is of one of the following forms:

$$\lambda_i < \lambda_{i+1} < \tau_i < \lambda_{i+2} < \tau_{i+1} < \lambda_{i+3} < \dots < \tau_{r-1} < \lambda_{r+1} < \tau_r < \tau_{r+1},$$

or

$$\tau_i < \tau_{i+1} < \lambda_{i+2} < \tau_{i+2} < \lambda_{i+3} < \tau_{i+3} < \dots < \lambda_r < \tau_r < \lambda_{r+1} < \lambda_{r+2}.$$

So, we can always assign pairs of consecutive elements of the form $\tau_j < \lambda_k$ or $\lambda_j < \tau_k$ to either sets and still the τ 's interlace the λ 's. We can assign enough such pairs to each set in order to get the correct sizes. Thus, the claim holds and the two sets satisfy the desired conditions. \square

Example 2.6 Let $k = 2$ and

$$X : \lambda_1 < \tau_1 < \lambda_2 < \lambda_3 < \tau_2 < \lambda_4 < \tau_3 < \tau_4 < \lambda_5 < \lambda_6 < \tau_5 < \tau_6 < \lambda_7 < \lambda_8.$$

Note that there are two τ -pairings in X . We want to partition X into two sets, B and C of λ 's interlaced by τ 's, where $|B| = 5 \geq 2 \cdot 2 + 1$ and $|C| = 9 \geq 2 \cdot 2 + 1$. One choice is to assign the first element in each pairing to B and the second element to C . Hence,

$$B : \quad \lambda_2 < \quad \quad \quad \tau_3 < \quad \quad \lambda_5 < \quad \quad \tau_5 < \quad \quad \lambda_7,$$

and

$$C : \quad \quad \quad \lambda_3 < \quad \quad \quad \tau_4 < \quad \quad \lambda_6 < \quad \quad \tau_6 < \quad \quad \lambda_8.$$

Finally

$$X' : \lambda_1 < \tau_1 < \tau_2 < \lambda_4.$$

Since B already has five elements, we assign the remaining two pairs in X' to C so that it has nine elements. That is,

$$B : \lambda_2 < \tau_3 < \lambda_5 < \tau_5 < \lambda_7,$$

and

$$C : \lambda_1 < \tau_1 < \lambda_3 < \tau_2 < \lambda_4 < \tau_4 < \lambda_6 < \tau_6 < \lambda_8.$$

Example 2.7 A concrete example is provided here. Let λ 's be 0 – 6, –5, –2, –1, 3 and 4 and let τ 's be –4, –3, 1 and 2. Then, using the above procedure we can get two sequences of λ 's and τ 's such that the τ 's strictly interlace the λ 's in each sequence. One such partition is:

$$-6, -\mathbf{4}, -2, \mathbf{1}, 3,$$

and

$$-5, -\mathbf{3}, -1, \mathbf{2}, 4,$$

where bold numbers are the τ 's.

2.2. Graph restrictions

In this section, we shall show that in addition to the restrictions on the λ - τ sequence given in Lemmas 2.2–2.4, there are restrictions related to the underlying graph. Throughout the remainder of this section, we assume that $A = [a_{ij}]$ is a real $n \times n$ symmetric matrix, the graph of A is a tree T and r and s are adjacent vertices in T (Figure 1). Removing the edge $\{r, s\}$ from T results in a graph with two connected components. We let V_r be the set of vertices in the connected component that contains r and V_s be the set of vertices of the other connected component. We let $\alpha_1, \alpha_2, \dots, \alpha_{i_r}$ be the vertices in V_r adjacent to r and $\beta_1, \beta_2, \dots, \beta_{i_s}$ be the vertices in V_s adjacent to s .

The following lemma, which relates the characteristic polynomial of A to the characteristic polynomials of $A(\{r, s\})$, $A(r)$ and $A(s)$, plays a key role. For a detailed proof of the case A a zero-one matrix see [9, Proposition 5.1.1].

LEMMA 2.8 Let $A = [a_{ij}]$ be a real symmetric $n \times n$ matrix whose graph is a tree T with vertices r and s adjacent. Let λ_i 's be the eigenvalues of A and τ_i 's be the eigenvalues of $A(\{r, s\})$. Then, the characteristic polynomial of A is

$$c_A(x) = -a_{rs}^2 c_{A[V_r \setminus \{r\}]} c_{A[V_s \setminus \{s\}]} + c_{A[V_r]} c_{A[V_s]} \tag{3}$$

and

$$\frac{\prod_{i=1}^n (x - \lambda_i)}{\prod_{i=1}^{n-2} (x - \tau_i)} = -a_{rs}^2 + \frac{c_{A[V_r]}}{c_{A[V_r \setminus \{r\}]}} \frac{c_{A[V_s]}}{c_{A[V_s \setminus \{s\}]}}. \tag{4}$$

Proof First observe that since T is a tree, each nonzero term in $\det(xI - A)$ containing a_{rs} as a factor also contains a_{sr} as a factor. The first term of (3) represents the terms of the

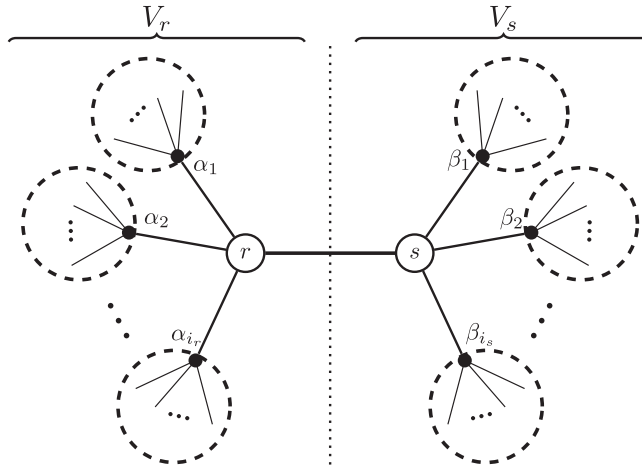


Figure 1. A tree T with adjacent vertices r and s , where other neighbors of r are $\alpha_1, \dots, \alpha_{i_r}$ and other neighbors of s are $\beta_1, \dots, \beta_{i_s}$.

polynomial that contain a_{rs} as a factor and the second term represents the terms that do not. Equation (4) is obtained by dividing both sides of (3) by $c_{A(\{r,s\})}$. \square

When the λ - τ sequence of A is nondegenerate, the following shows that the cardinalities of V_r and V_s are upper bounds on the number of τ -pairings of this sequence.

LEMMA 2.9 *Let A be an $n \times n$ real symmetric matrix with the property that its graph is a tree T , vertices r and s are adjacent in T and the λ - τ sequence of $(A, A(\{r, s\}))$ is nondegenerate. If there are exactly k τ -pairings in the λ - τ sequence of $(A, A(\{r, s\}))$, then $|V_r|, |V_s| > k$.*

Proof We claim that for each τ -pairing, one of the τ 's is an eigenvalue of $A[V_r \setminus \{r\}]$ and the other one is an eigenvalue of $A[V_s \setminus \{s\}]$. Suppose to the contrary that both of the τ 's in the pairing $\lambda_{i+1} < \tau_i < \tau_{i+1} < \lambda_{i+2}$ belong to $A[V_s \setminus \{s\}]$. Then, τ_i and τ_{i+1} are also eigenvalues of $A(s)$, which by first order Cauchy interlacing inequalities should interlace the λ 's. That is, there is an eigenvalue λ of A , such that $\tau_i < \lambda < \tau_{i+1}$. This contradicts our assumption that τ_i and τ_{i+1} form a pairing. Hence, each of the subgraphs $T[V_r \setminus \{r\}]$ and $T[V_s \setminus \{s\}]$ has one vertex for each τ -pairing and we conclude that $|V_r|, |V_s| > k$. \square

Example 2.10 Let T be as in Figure 2 and A be a symmetric matrix with graph T . Then, by Lemma 2.9 the λ - τ sequence of $(A, A(\{1, 2\}))$ is not of the form $\lambda_1 < \lambda_2 < \tau_1 < \tau_2 < \lambda_3 < \lambda_4$, since $|V_1| = 1$ and there is a τ -pairing.

The following lemma simply shows that for a rational function whose roots are all simple, there is a sufficiently small vertical shift such that it does not change the number of roots between any two poles and all the new roots are distinct from the old roots and the poles of the original function.

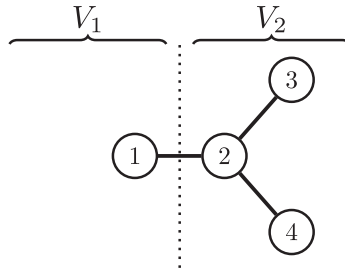


Figure 2. A star on four vertices.

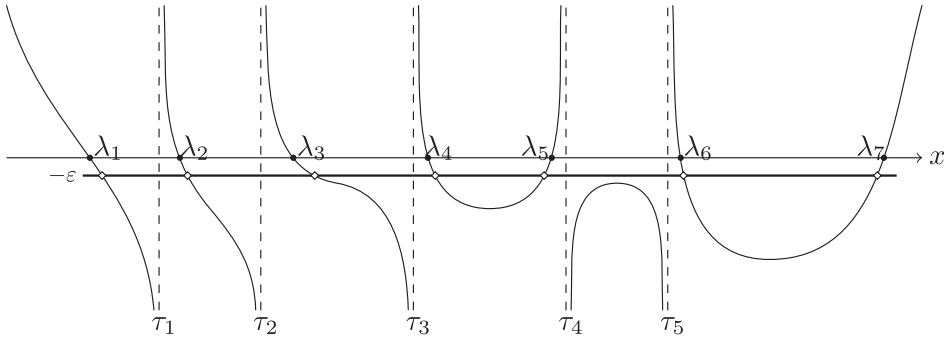


Figure 3. An example of the function f defined by (5) and a suitable choice of small enough ε .

LEMMA 2.11 *Let*

$$f(x) = \frac{\prod_{i=1}^n (x - \lambda_i)}{\prod_{i=1}^{n-2} (x - \tau_i)}, \tag{5}$$

where λ_i 's and τ_i 's satisfy the strict second-order Cauchy interlacing inequalities and the λ - τ sequence is nondegenerate. Then, for sufficiently small ε the function $f(x) + \varepsilon$ has exactly n distinct real roots, say $\mu_1, \mu_2, \dots, \mu_n$, where $\mu_i \neq \tau_j$ for all i and j (Figure 3). Moreover, in the μ - τ sequence, the μ 's are exactly in the same position as λ 's in the λ - τ sequence. That is, the τ_i 's interlace the μ_i 's in the same fashion that they interlace the λ_i 's.

Proof Since the λ - τ sequence is nondegenerate, $f(x)$ has exactly n roots $\lambda_1, \lambda_2, \dots, \lambda_n$ and all of them are simple roots. Furthermore, $f(x)$ is a continuous and differentiable function around its roots, so for each $i = 1, 2, \dots, n$ there exist $\delta_i > 0$ such that $f'(x) \neq 0$ on the interval $[\lambda_i - \delta_i, \lambda_i + \delta_i]$. Let $\varepsilon = \frac{1}{2} \min\{|f(\lambda_i \pm \delta_i)| : i = 1, \dots, n\}$. \square

Note that if $f(x)$ does not have any negative local extreme values, then ε can be chosen arbitrarily large. But, since there is at least one λ -pairing, it can be shown that there is at least one negative local extreme value for the function f . Hence, the choice of ε will be restricted to $(0, m)$, where m is the maximum of these negative local extreme values.

3. The λ - τ structured inverse eigenvalue problem for trees

Recall that there is a $\varepsilon > 0$ such that τ_i 's interlace the n real roots of $f(x) + \varepsilon$ in the same way that they interlace λ_i 's. Below the (r, s) entry of A is chosen such that $0 \leq a_{rs} \leq \sqrt{\varepsilon}$. Theorem 3.1 below shows that any such choice of a_{rs} is sufficient for solving the λ - τ problem for trees, provided the necessary conditions are satisfied. In other words, assuming nondegeneracy of the λ - τ sequence, the only constraints to solve a λ - τ structured inverse eigenvalue problem where the given graph is a tree and the vertices to be deleted are adjacent, are the second-order Cauchy interlacing inequalities (Proposition 2.1) and the combinatorial restrictions (Lemma 2.9). Now we are ready to present and prove the main theorem for trees.

THEOREM 3.1 *Let T be a tree with vertices $1, 2, \dots, n$ such that its vertices r and s are adjacent and $\lambda_1, \dots, \lambda_n, \tau_1, \dots, \tau_{n-2}$ be real numbers satisfying*

$$\lambda_i < \tau_i < \lambda_{i+2}, \tag{6}$$

and

$$\tau_i \neq \lambda_{i+1}, \tag{7}$$

for all $i = 1, \dots, n - 2$. Furthermore, assume that k τ -pairings occur and $T[V_r \setminus \{r\}]$ and $T[V_s \setminus \{s\}]$ each have at least k vertices. Then, there is a symmetric matrix $A = [a_{ij}]$ with graph T and eigenvalues $\lambda_1, \dots, \lambda_n$ such that $A(\{r, s\})$ has eigenvalues $\tau_1, \dots, \tau_{n-2}$.

Proof Let T be a tree as in Figure 1 and $f(x)$ be defined by (5). By Lemma 2.11 there exists an $\varepsilon > 0$ such that $g(x) = f(x) + \varepsilon$ has n distinct real zeros. Let $a_{rs} = \sqrt{\varepsilon}$ and $\mu_1, \mu_2, \dots, \mu_n$ be the roots of $g(x)$. For small enough $\varepsilon > 0$ the τ 's interlace the μ 's in the same way that τ 's interlace λ 's. Let X be the set of these μ 's and τ 's. Then, X is nondegenerate with exactly k τ -pairings, $|V_r|$ and $|V_s|$ are positive integers such that $|V_r| + |V_s| = n$ and $|V_r|, |V_s| > k$. Thus, by Lemma 2.5 X can be partitioned into two sets X_1, X_2 such that X_1 has $|V_r|$ μ 's and $|V_r| - 1$ τ 's and X_2 set has $|V_s|$ μ 's and $|V_s| - 1$ τ 's. Furthermore, in each set the τ 's and the μ 's satisfy first order Cauchy interlacing inequalities.

By Lemma 2.1 of [6], there are real symmetric matrices $A[V_r]$ and $A[V_s]$ such that graph of $A[V_r]$ is $T[V_r]$ and graph of $A[V_s]$ is $T[V_s]$. The set X_1 consists of the eigenvalues of $A[V_r]$ and $A[V_r \setminus \{r\}]$ and the set X_2 consists of the eigenvalues of $A[V_s]$ and $A[V_s \setminus \{s\}]$.

Now let $A = (A[V_r] \oplus A[V_s]) + a_{rs}(E_{rs} + E_{sr})$, where $E_{rs} + E_{sr}$ represents the matrix with 1's in the positions corresponding to the edge $\{r, s\}$ and zeros elsewhere. By Lemma 2.8, the eigenvalues of A are λ_i 's and the eigenvalues of $A(\{r, s\})$ are τ_i 's.

$$A = \left[\begin{array}{c|c} A[V_r] & O \\ \hline O & A[V_s] \end{array} \right] \begin{array}{c} a_{rs} \\ a_{rs} \end{array} \quad \square$$

We note that if $r = 1$ and $s = 2$, then by reordering the rows and the columns as $(1, 2, \alpha_1, \dots, \alpha_i, \beta_1, \dots, \beta_i)$, A has the form:

$$A = \left[\begin{array}{c|c|c|c} (A_\alpha)_{11} & a_{12} & A_\alpha[1, 1] & 0 \ \dots \ 0 \\ \hline a_{12} & (A_\beta)_{11} & 0 \ \dots \ 0 & A_\beta[1, 1] \\ \hline A_\alpha(1, 1] & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} & A_\alpha(1) & O \\ \hline \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} & A_\beta(1, 1] & O & A_\beta(1) \end{array} \right],$$

where $A_\alpha = A[V_1]$ and $A_\beta = A[V_2]$.

Example 3.2 Suppose we want to find a real symmetric matrix A whose graph is the tree T in Figure 4, such that its eigenvalues are $-6, -5, -2, -1, 3, 4$ and 6 and the eigenvalues of $A(\{3, 4\})$ are $-4, -3, 1, 2$ and 5 . There are two τ -pairings and they interlace the three λ -pairings. Note that $|V_3|, |V_4| > 2$. So, Theorem 3.1 guarantees the existence of such matrix A . To construct this matrix, we choose $a_{34} = 1$ and find the roots μ_i of $f(x) + 1$ where f is defined by (5). Partition the μ - τ sequence into two sequences similar to the partitioning in Example 2.7. Following the algorithm given in the proof of Theorem 3.1, we find

$$A \simeq \left[\begin{array}{ccc|cccc} -4 & 0 & 2.292 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2.856 & 0 & 0 & 0 & 0 \\ 2.292 & 2.856 & -1.699 & \mathbf{1} & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{1} & -0.3008 & 1.620 & 4.180 & 0 \\ 0 & 0 & 0 & 1.620 & 5 & 0 & 0 \\ 0 & 0 & 0 & 4.180 & 0 & 0.2033 & 2.399 \\ 0 & 0 & 0 & 0 & 0 & 2.399 & -1.203 \end{array} \right].$$

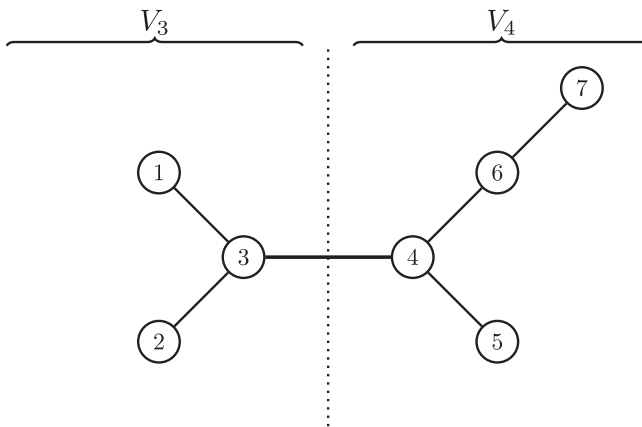


Figure 4. A tree T on 7 vertices with adjacent vertices 3 and 4.

Note that if we let $a_{3,4} = 0.1$, we get a different matrix

$$A \simeq \left[\begin{array}{ccc|cccc} -4 & 0 & 2.366 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2.898 & 0 & 0 & 0 & 0 \\ 2.366 & 2.898 & -1.997 & \mathbf{0.1} & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{0.1} & -0.002686 & 1.581 & 4.183 & 0 \\ 0 & 0 & 0 & 1.581 & 5 & 0 & 0 \\ 0 & 0 & 0 & 4.183 & 0 & 0.2001 & 2.4 \\ 0 & 0 & 0 & 0 & 0 & 2.4 & -1.2 \end{array} \right].$$

It is easy to check that the graph of A is T and the eigenvalues of A and $A(\{3, 4\})$ are as desired. Note that the approximations are because of machine error and also approximations in finding the roots of the polynomials during the algorithm. Furthermore, note that we can choose all the off-diagonal entries corresponding to an edge of the graph to be positive.

4. The λ - τ structured inverse eigenvalue problem for connected graphs

It is natural to ask if there is a result analogous to Theorem 3.1 for general connected graphs. First, note that a connected graph which is not a tree has at least 3 vertices. So, for the rest of this section we can safely assume that $n \geq 3$. Here we use a similar approach to the one in [6] using the Implicit Function Theorem, the Duarte property and a property similar to the Strong-Arnold hypothesis to give an affirmative answer to this question. Let A be a matrix whose graph is a tree T on n vertices $1, 2, \dots, n$, with v and w adjacent. Let $T(w)$ denote the submatrix obtained from T by deleting the vertex w and $T_v(w)$ denote the connected component of $T(w)$ which contains v . Also let $A(w)$ and $A_v(w)$ denote the submatrices of A corresponding to $T(w)$ and $T_v(w)$, respectively. Here, we quote the definition of the Duarte property from [6]. For a precise definition of other concepts see [6].

Let A be a real symmetric matrix. If $G(A)$ has just one vertex, then A has the Duarte-property with respect to w . If $G(A)$ has more than one vertex, then A has the Duarte-property with respect to w provided the eigenvalues of $A(w)$ strictly interlace those of A and for each neighbor v of w , $A_v(w)$ has the Duarte-property with respect to the vertex v .

Note that by construction, the matrices $A[V_r]$ and $A[V_s]$ in the Theorem 3.1 can be taken to have the Duarte property with respect to vertices r and s , respectively. Let $A = A[V_r] \oplus A[V_s]$, $\mathbf{x} = (x_1, x_2, \dots, x_{2n-2})$ and $\mathbf{y} = (y_1, y_2, \dots, y_p)$, where x_i 's and y_j 's are real variables and $p = \frac{n^2-3n+4}{2}$.

Let $M(\mathbf{x}, \mathbf{y})$ be a matrix obtained from A by replacing diagonal entries by $2x_i$, $1 \leq i \leq n$, nonzero off-diagonal entries by x_{n+i} , $1 \leq i \leq n - 2$ and zero off-diagonal entries by y_j , $1 \leq j \leq p$. Note that the entry corresponding to the edge $\{1, 2\}$ is now replaced by some y_j . Also define $N(\mathbf{x}, \mathbf{y}) := (M(\mathbf{x}, \mathbf{y}))(\{r, s\})$. We abbreviate $M(\mathbf{x}, \mathbf{y})$ and $N(\mathbf{x}, \mathbf{y})$ by M and N , respectively. Let $\mathbf{b} = (b_1, \dots, b_p)$. For a function $f(\mathbf{x}, \mathbf{y})$ and a matrix $A = M(a_1, \dots, a_{2n-1}, b_1, \dots, b_p)$ we denote $f(a_1, \dots, a_{2n-1}, b_1, \dots, b_p)$ by $f(A, \mathbf{b})$. Similarly, $\text{Jac}(f)|_{(A, \mathbf{b})}$ denotes the Jacobian matrix of f where it is evaluated at $(\mathbf{x}, \mathbf{y}) = (a_1, \dots, a_{2n-1}, b_1, \dots, b_p)$.

Define $g : \mathbb{R}^{2n-2} \times \mathbb{R}^p \rightarrow \mathbb{R}^{2n-2}$ by

$$g(\mathbf{x}, \mathbf{y}) = (c_0, c_1, \dots, c_{n-1}, d_0, d_1, \dots, d_{n-3}),$$

where the c_i 's and d_i 's are the nonleading coefficients of the characteristic polynomials of M and N , respectively. We want to show that if $g(A, \mathbf{0}) = (\mathbf{c}, \mathbf{d}) \in \mathbb{R}^{2n-2}$ for some 'generic' $(A, \mathbf{0}) \in \mathbb{R}^{2n-2} \times \mathbb{R}^p$, then for any sufficiently small perturbations $\boldsymbol{\varepsilon} \in \mathbb{R}^p$ there is an adjustment of $A \in \mathbb{R}^{2n-2}$, namely \hat{A} , such that $g(\hat{A}, \boldsymbol{\varepsilon}) = (\mathbf{c}, \mathbf{d})$. In other words, if the coefficients of the characteristic polynomials of A and $A(\{r, s\})$ are given by \mathbf{c} and \mathbf{d} , respectively, then any superpattern of A has a realization with the same characteristic polynomial.

It is hard to work with partial derivatives of g . Using Newton's identities, we introduce a function f such that there exist a differentiable, invertible function h with $f \circ h = g$. Thus, similar to g , if $f(h(A, \mathbf{0})) = (\mathbf{a}, \mathbf{b})$, then there is a matrix \hat{A} such that $f(h(\hat{A}, \boldsymbol{\varepsilon})) = (\mathbf{a}, \mathbf{b})$.

Define the function $f : \mathbb{R}^{2n-2} \times \mathbb{R}^p \rightarrow \mathbb{R}^{2n-2}$ by

$$f(\mathbf{x}, \mathbf{y}) = \left(\frac{\text{tr}M}{2}, \frac{\text{tr}M^2}{4}, \dots, \frac{\text{tr}M^n}{2n}, \frac{\text{tr}N}{2}, \frac{\text{tr}N^2}{4}, \dots, \frac{\text{tr}N^{n-2}}{2(n-2)} \right). \tag{8}$$

Let $\text{Jac}_x(f)$ be the matrix obtained from the Jacobian of f by deleting the columns corresponding to derivatives of f with respect to y_i 's. We will show that $\text{Jac}_x(f)$ evaluated at $(A, \mathbf{0})$ is nonsingular. The same calculations as in Lemma 3.1 of [6] yield the following:

LEMMA 4.1 *Let M and N be as above and (i, j) be a nonzero position of M with corresponding variable x_t . Then,*

- (a) $\frac{\partial}{\partial x_t} (\text{tr}M^k) = 2kM_{ij}^{k-1}$ and
- (b) $\frac{\partial}{\partial x_t} (\text{tr}N^k) = \begin{cases} 2kN_{ij}^{k-1} & \text{if neither } i \text{ nor } j \text{ is } 1 \text{ or } 2 \\ 0 & \text{otherwise.} \end{cases}$

Notation

- For simplicity from now on let $r = 1$ and $s = 2$. Furthermore, assume that 1 is the first vertex of V_1 and 2 is the first vertex of V_2 . Then for example, $A[V_2 \setminus \{2\}] = A[V_2](1)$
- \tilde{C} is a matrix obtained from a matrix C by appending two zero rows on top of it and then two zero columns to left of the new matrix. That is,

$$\tilde{C} = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & & \\ 0 & 0 & & & \mathbf{C} \end{array} \right].$$

- A '*' as an entry of a matrix means a real number whose value is not known.
- for two matrices A and B of the same size, $A \circ B$ denotes the entry-wise product of A and B , also known as the Schur product, or Hadamard product. That is, $(A \circ B)_{ij} = A_{ij}B_{ij}$.
- For two $n \times n$ matrices A and B , $[A, B] := AB - BA$ is the commutator of A and B .

Using Lemma 4.1 we can see the following.

COROLLARY 4.2 *Let f and A be defined as above. Then,*

$$\text{Jac}_x(f) \Big|_{(A, \mathbf{0})} = \begin{bmatrix} I_{i_1 j_1} & \cdots & I_{i_{n-1} j_{n-1}} & I_{11} & \cdots & I_{nn} \\ A_{i_1 j_1} & \cdots & A_{i_{n-1} j_{n-1}} & A_{11} & \cdots & A_{nn} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{i_1 j_1}^{n-1} & \cdots & A_{i_{n-1} j_{n-1}}^{n-1} & A_{11}^{n-1} & \cdots & A_{nn}^{n-1} \\ \hline \tilde{I}_{i_1 j_1} & \cdots & \tilde{I}_{i_{n-1} j_{n-1}} & \tilde{I}_{11} & \cdots & \tilde{I}_{nn} \\ \tilde{B}_{i_1 j_1} & \cdots & \tilde{B}_{i_{n-1} j_{n-1}} & \tilde{B}_{11} & \cdots & \tilde{B}_{nn} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \tilde{B}_{i_1 j_1}^{n-3} & \cdots & \tilde{B}_{i_{n-1} j_{n-1}}^{n-3} & \tilde{B}_{11}^{n-3} & \cdots & \tilde{B}_{nn}^{n-3} \end{bmatrix}.$$

Note that this is a $(2n - 2) \times (2n - 2)$ matrix and in order to be nonsingular it suffices to show that it has full row rank.

For convenience, we mention Lemmas 1.1 and 2.2 of [6] here.

LEMMA 4.3 *Let A be an $m \times m$ matrix, B be an $n \times n$ matrix and X be an $m \times n$ matrix such that $AX = XB$. Then the following hold:*

- (a) *If A and B do not have a common eigenvalue, then $X = O$.*
- (b) *If $X \neq O$ and A and B share exactly one common eigenvalue, then each nonzero column of X is a generalized eigenvector of A corresponding to the common eigenvalue.*

LEMMA 4.4 *Let A have the Duarte-property with respect to the vertex w , $G(A)$ be a tree T and X be a symmetric matrix such that*

- (a) $I \circ X = O$,
- (b) $A \circ X = O$,
- (c) $[A, X](w) = O$.

Then $X = O$.

The following theorem shows that the matrix $A = A[V_r] \oplus A[V_s]$ constructed in the proof of Theorem 3.1 is a ‘generic’ matrix. That is, the Jacobian of the function f defined by (8) evaluated at A is nonsingular.

THEOREM 4.5 *Let T be a tree on n vertices $\{1, 2, \dots, n\}$, where $e = \{r, s\}$ is an edge of T . Let A be a real symmetric matrix whose graph is $T \setminus e$, the function f be defined by (8) and $B = A(\{r, s\})$. If $A[V_r]$ has the Duarte-property with respect to vertex r and $A[V_s]$ has the Duarte-property with respect to vertex s , then $\text{Jac}_x(f) \Big|_{(A, \mathbf{0})}$ is nonsingular.*

Proof Let $\alpha = (\alpha_1, \dots, \alpha_{2n-2})$ and assume that $\alpha^T \text{Jac}_x(f) \Big|_{(A,0)} = O$, that is,

$$\sum_{i=1}^{2n-2} \alpha_i \text{Jac}(f)_k = O, \tag{9}$$

where $\text{Jac}(f)_k$ denotes the k^{th} row of $\text{Jac}_x(f) \Big|_{(A,0)}$. We want to show that $\alpha = \mathbf{0}$.

Let $X = \alpha_1 I + \alpha_2 A + \dots + \alpha_n A^{n-1} + \alpha_{n+1} \widetilde{I} + \alpha_{n+2} \widetilde{B}^1 + \dots + \alpha_{2n-2} \widetilde{B}^{n-3}$. Note that each column of $\text{Jac}_x(f) \Big|_{(A,0)}$ is evaluated only at a diagonal or a nonzero off-diagonal position of A . Thus $\alpha^T \text{Jac}_x(f) \Big|_{(A,0)} = O$ if and only if

- all diagonal entries of X are zero and
- wherever A has a nonzero entry, X has a zero entry.

That is, $X \circ A = O$ and $X \circ I = O$.

We first show that $X = O$. Let

$$p(x) = \sum_{i=1}^n \alpha_i x^{i-1}, \text{ and } q(x) = \sum_{j=n+1}^{2n-1} \alpha_j x^{j-(n+1)}.$$

Then $X = p(A) + \widetilde{q(B)}$ and $\text{Jac}_x(f) \Big|_{(A,0)}$ has full row rank if $p(x)$ and $q(x)$ are both zero polynomials. Since $[A, p(A)] = O$, $[A, X] = [A, \widetilde{q(B)}]$. Also since $A(\{r, s\}) = B$, $[A, \widetilde{q(B)}](\{r, s\}) = O$. Hence, $[A, X](\{r, s\}) = O$.

Reorder rows and columns of A so that

$$A = \left[\begin{array}{c|c|c} C & x & O \\ \hline x^T & * & \\ \hline O & \begin{array}{c|c} * & y^T \\ \hline y & D \end{array} & \end{array} \right],$$

where x and y correspond to vertices r and s , respectively. Then

$$\widetilde{q(B)} = \left[\begin{array}{ccc|ccc} & & & 0 & & \\ & & & \vdots & & \\ & & & 0 & & \\ \hline & q(C) & & 0 & & O \\ \hline 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \hline & & & 0 & & & \\ & O & & 0 & & & \\ & & & \vdots & & & \\ & & & 0 & & & \\ & & & & q(D) & & \end{array} \right].$$

By direct calculations we have

$$[A, \widetilde{q(B)}] = \left[\begin{array}{ccc|ccc} & & & * & & \\ & & & \vdots & & \\ & & & * & & \\ \hline & O & & 0 & & O \\ \hline * & \dots & * & 0 & 0 & * & \dots & * \\ \hline & & & & 0 & * & \dots & * \\ & & & & * & & & \\ & O & & & \vdots & & & \\ & & & & * & & & \\ & & & & & O & & \end{array} \right].$$

Recall that, the (1, 1) block of the above 2×2 block matrix corresponds to the indices in V_r and its (2, 2) block corresponds to the indices in V_s . It follows that $[A[V_r], X[V_r]](r) = O$ and $[A[V_s], X[V_s]](s) = O$. Recall that the graphs of $A[V_1]$ and $A[V_2]$ are trees and these matrices are chosen to have the Duarte property with respect to the vertices 1 and 2, respectively. Thus, by Lemma 4.4 $X[V_r] = O$ and $X[V_s] = O$. So far, it is shown that X has the following form:

$$X = \left[\begin{array}{c|c} O & X_1 \\ \hline X_2 & O \end{array} \right],$$

where the blocks are conformally partitioned as A . But X is a polynomial in A and \widehat{B} , hence X_1 and X_2 are also zero, hence, $X = O$. Thus, $p(A) = -\widetilde{q(B)}$. Let $Y = p(A) = -\widetilde{q(B)}$. Note that,

$$AY = Ap(A) = -A\widetilde{q(B)} = \left[\begin{array}{ccc|ccc} & & & 0 & & \\ & & & \vdots & & \\ & & & 0 & & \\ \hline & Cq(C) & & 0 & & O \\ \hline * & \dots & * & 0 & 0 & * & \dots & * \\ \hline & & & & 0 & * & \dots & * \\ & & & & 0 & & & \\ & & & & \vdots & & & \\ & O & & & 0 & & & \\ & & & & & Dq(D) & & \end{array} \right],$$

and

$$YA = p(A)A = -\widetilde{q(B)}A = \left[\begin{array}{ccc|ccc} & & & * & & \\ & q(C)C & & \vdots & & \\ & & & * & & \\ \hline 0 & \dots & 0 & 0 & & \\ & & & & 0 & 0 \dots 0 \\ \hline & & & & * & \\ & O & & & \vdots & \\ & & & & * & \\ & & & & & q(D)D \end{array} \right].$$

Since $Ap(A) = p(A)A$, the stars are all zero. That is, $AY = Y\widetilde{B}$. Hence, by part (a) of Lemma 4.3 either $Y = O$, or A and \widetilde{B} have a common eigenvalue. If $Y = O$ we are done. Otherwise, since A and B have no common eigenvalue, A and \widetilde{B} both have an eigenvalue 0 and the multiplicity of it in A is 1. Suppose Y_j is a nonzero column of Y . Then, by part (b) of Lemma 4.3, Y_j is a generalized eigenvector of A corresponding to 0. But A has distinct eigenvalues, hence Y_j is an eigenvector of A corresponding to 0. Note that since $Y = -\widetilde{q(B)}$, $Y_j = [* \dots * 0 | 0 * \dots *]^T$, where the blocks are the same size as C and D . The form of A and Y_j imply that the vector $Y_j(\{r, s\})$ is a nonzero eigenvector of B corresponding to 0. This leads to a contradiction that A and B have a common eigenvalue. Thus $Y = O$.

Since $Y = O$, $p(A) = O$ and $q(B) = O$. Note that $p(x)$ is a polynomial of degree at most $n - 1$. Since A has n distinct eigenvalues, its minimal polynomial has degree n . Thus, $p(x)$ is the zero polynomial. Similarly, $q(x)$ is the zero polynomial. So $\text{Jac}_x(f)|_{(A,0)}$ is nonsingular. \square

Now we are ready to prove an analogue to Theorem 3.1 for connected graphs.

We state below a version of the Implicit Function Theorem for convenience, to prove our main result (see [10]).

THEOREM 4.6 (Implicit Function Theorem) *Let $F : \mathbb{R}^{s+r} \rightarrow \mathbb{R}^s$ be a continuously differentiable function on an open subset U of \mathbb{R}^{s+r} defined by*

$$F(\mathbf{x}, \mathbf{y}) = (F_1(\mathbf{x}, \mathbf{y}), F_2(\mathbf{x}, \mathbf{y}), \dots, F_s(\mathbf{x}, \mathbf{y})),$$

where $\mathbf{x} = (x_1, \dots, x_s) \in \mathbb{R}^s$ and $\mathbf{y} \in \mathbb{R}^r$. Let (\mathbf{a}, \mathbf{b}) be an element of U with $\mathbf{a} \in \mathbb{R}^s$ and $\mathbf{b} \in \mathbb{R}^r$, and \mathbf{c} be an element of \mathbb{R}^s such that $F(\mathbf{a}, \mathbf{b}) = \mathbf{c}$. If

$$\left[\frac{\partial F_i}{\partial x_j} \Big|_{(\mathbf{a}, \mathbf{b})} \right]$$

is nonsingular, then there exist an open neighborhood V containing \mathbf{a} and an open neighborhood W containing \mathbf{b} such that $V \times W \subseteq U$ and for each $\mathbf{y} \in W$ there is an $\mathbf{x} \in V$ with $F(\mathbf{x}, \mathbf{y}) = \mathbf{c}$.

THEOREM 4.7 *Let G be a connected graph on n vertices $1, 2, \dots, n$ with vertices 1 and 2 adjacent in G . Furthermore, assume that G has a spanning tree T containing the edge $\{1, 2\}$ and a partition $V_1 \cup V_2$ of its vertices with $|V_1|, |V_2| > k$ such that V_i contains vertex*

i for $i = 1, 2$. Let $\lambda_1, \dots, \lambda_n, \tau_1, \dots, \tau_{n-2}$ be real numbers satisfying

$$\lambda_i < \tau_i < \lambda_{i+2}, \tag{10}$$

$$\tau_i \neq \lambda_{i+1}, \tag{11}$$

for all $i = 1 \dots, n - 2$. If k τ -pairings occur in the given λ - τ sequence, then there is a real symmetric matrix $A = [a_{ij}]$ with graph G and eigenvalues $\lambda_1, \dots, \lambda_n$ such that eigenvalues of $A(\{1, 2\})$ are $\tau_1, \dots, \tau_{n-2}$.

Proof Consider the spanning tree T of G . Theorem 3.1 implies that there exists an $A = (A[V_1] \oplus A[V_2]) + a_{12}(E_{12} + E_{21}) \in S(T)$ such that A has eigenvalues $\lambda_1, \dots, \lambda_n$, $A(\{1, 2\})$ has eigenvalues $\tau_1, \dots, \tau_{n-2}$ and $A[V_1], A[V_2]$ have the Duarte-property with respect to vertex 1 and 2, respectively. By Theorem 4.5, the Jacobian matrix of the function f evaluated at A is nonsingular.

Let c and d be the vectors of nonleading coefficients of the characteristic polynomials of A and $A(\{1, 2\})$, respectively.

Letting $a = (a_1, \dots, a_n, a_{n+1}, a_{2n-2})$ be the assignment of the x_j 's corresponding to A we see that $g(a, 0, 0, \dots, 0) = (c, d)$. Since a_{n+1}, \dots, a_{2n-2} are nonzero, there is an open neighborhood U of $(a, 0, \dots, 0)$ each of whose elements has no zeros in the same $n - 2$ positions. By the Implicit Function Theorem 4.6, there is an open neighborhood V of a and an open neighborhood W of $\mathbf{0}$ such that $V \times W \subseteq U$ and for each $y \in W$ there is an $x \in V$ such that $F(x, y) = (c, d)$. Take y to be a vector in W with no zero entries corresponding to the edges of G . Then the (x, y) satisfying $F(x, y) = (c, d)$ corresponds to a matrix $\widehat{A} \in S(G)$ such that the λ 's are the eigenvalues of \widehat{A} and the τ 's are the eigenvalues of $\widehat{A}(\{1, 2\})$. \square

Remark 4.8 Note that in the proof of Theorem 4.5, no conditions on the choice of a_{rs} are placed. For example, a_{rs} can be zero. Similarly, it could remain zero in the proof of Theorem 4.7. Hence, to prove Theorem 3.1, one could prove it for the forest obtained by deleting the edge $\{1, 2\}$, that is, by letting $a_{12} = 0$ in A . Then use Theorem 4.7 to extend it to the original tree, that is, a superpattern of the obtained forest.

5. The case when the two removed vertices are not adjacent

In this section, assume T is a tree on vertices $1, 2, \dots, n$, where the vertices 1 and 2 are not adjacent (Figure 5). Let

$$\alpha = \left\{ v \mid \begin{array}{l} \text{the path from } v \text{ to } 1 \text{ does not contain } \\ 2 \text{ and the path from } v \text{ to } 2 \text{ contains } 1. \end{array} \right\}$$

$$\beta = \left\{ v \mid \begin{array}{l} \text{the path from } v \text{ to } 2 \text{ does not contain } \\ 1 \text{ and the path from } v \text{ to } 1 \text{ contains } 2. \end{array} \right\}$$

$$\gamma = \left\{ v \mid \begin{array}{l} \text{the path from } v \text{ to } 2 \text{ does not contain } 1 \text{ and the } \\ \text{path from } v \text{ to } 1 \text{ doesn't contain } 2. \end{array} \right\}$$

First note that in this case a variation of Lemma 2.9 holds.

LEMMA 5.1 *Let A be an $n \times n$ real symmetric matrix whose graph is a tree T , as above. If there are exactly k τ -pairings in the eigenvalues of A and $A(\{1, 2\})$, then $|\alpha|, |\beta| > k - |\gamma|$.*

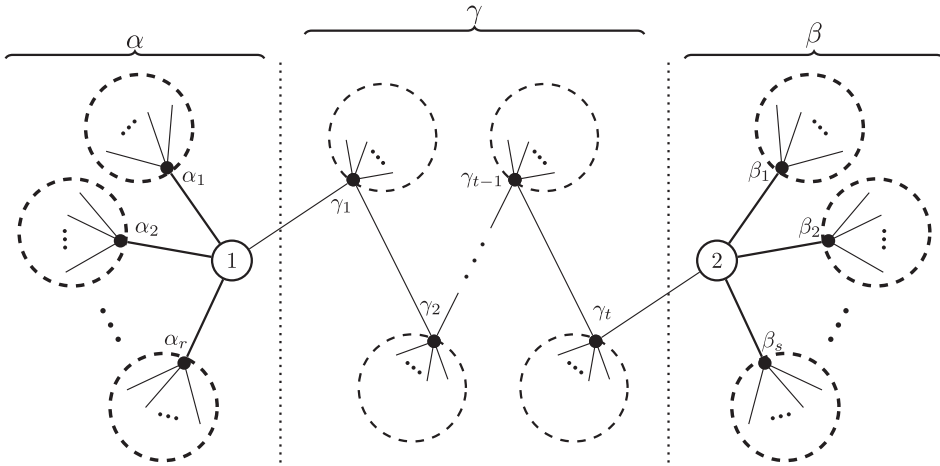


Figure 5. A tree T with nonadjacent vertices r and s .

Proof The proof is similar to that of Lemma 2.9. That is, both of the τ 's in a τ -pairing cannot both be eigenvalues of either $T[\alpha \setminus \{1\}]$ or $T[\beta \setminus \{2\}]$. Thus, each τ is an eigenvalue of $T[\alpha \setminus \{1\}]$, $T[\gamma]$ or $T[\beta \setminus \{2\}]$. In other words, for each τ -pairing $\lambda_{i+1}\tau_i < \tau_{i+1} < \lambda_{i+1}$, if neither τ_i nor τ_{i+1} belongs to $T[\beta \setminus \{2\}]$, then it belongs to $T[\gamma]$ or $T[\alpha \setminus \{1\}]$. Hence, $|\alpha| - 1 + |\gamma| \geq k$ and similarly $|\beta| - 1 + |\gamma| \geq k$. \square

In order to solve a λ - τ problem for a tree where the deleted vertices are not adjacent, we break T into two trees by deleting an edge in γ and solve two λ - μ problems similar to the ones of the Theorem 3.1. Then we show that this solution is generic in the same sense as in Theorem 4.5. Finally, we want to insert the deleted edge back to the tree and use the implicit function theorem to show that there is a solution. If one cannot divide γ into two parts γ_1, γ_2 such that $T[\alpha \cup \gamma_1]$ and $T[\beta \cup \gamma_2]$ are connected and each has at least $k + 1$ vertices, then our method does not work. Hence, we assume that such a partition of γ exists:

Assumption 5.2 There exist $\gamma_1, \gamma_2 \subseteq \gamma$ such that $\gamma_1 \cup \gamma_2 = \gamma$, $\gamma_1 \cap \gamma_2 = \emptyset$ and $T[\alpha \cup \gamma_1]$ and $T[\beta \cup \gamma_2]$ are connected and each has at least $k + 1$ vertices.

THEOREM 5.3 Let T be a tree on n vertices $1, 2, \dots, n$ such that r and s are not adjacent and $\lambda_1, \dots, \lambda_n, \tau_1, \dots, \tau_{n-2}$ real numbers satisfying

$$\lambda_i < \tau_i < \lambda_{i+2} \tag{12}$$

$$\tau_i \neq \lambda_{i+1}, \tag{13}$$

for all $i = 1 \dots, n-2$. Furthermore, assume that there are k τ -pairings and Assumption 5.2 holds. Then there is a symmetric matrix $A = [a_{ij}]$ with graph T and eigenvalues $\lambda_1, \dots, \lambda_n$ such that $A(\{r, s\})$ has eigenvalues $\tau_1, \dots, \tau_{n-2}$.

Proof As T is a tree, there exists an edge $\{u, v\}$ on the path from r to s that divides γ into the two sets γ_r and γ_s . Partition the λ - τ sequence of $(A, A(\{r, s\}))$ into two sets X_r and X_s as in Lemma 2.5. By Theorem 3.1 there are matrices $A[\alpha \cup \gamma_r]$, $A[\beta \cup \gamma_s]$ such that X_r consists of the eigenvalues of $A[\alpha \cup \gamma_r]$ and $A[\alpha \setminus \{r\} \cup \gamma_r]$ and X_s consists of the eigenvalues of $A[\beta \cup \gamma_s]$ and $A[\beta \setminus \{s\} \cup \gamma_s]$. Furthermore, $G(A[\alpha \cup \gamma_r]) = T[\alpha \cup \gamma_r]$ and $G(A[\beta \cup \gamma_s]) = T[\beta \cup \gamma_s]$. Note that $A[\alpha \cup \gamma_r]$ can be taken to have the Duarte property with respect to vertex r and $A[\beta \cup \gamma_s]$ can be taken to have the Duarte property with respect to vertex s .

Note that $a_{rs} = 0$. Let

$$A = \begin{array}{c} \begin{array}{c} u \quad v \\ \left[\begin{array}{c|c} A[\alpha \cup \gamma_1] & 0 \\ \hline 0 & A[\beta \cup \gamma_2] \end{array} \right] \begin{array}{c} u \\ v \end{array} \end{array} \end{array}$$

Let $G(A, 0) = (\mathbf{c}, \mathbf{d})$, where 0 comes from the (u, v) entry of A . By Theorem 4.5, the Jacobian of the function f defined in (8) evaluated at A is non-singular. Hence, for sufficiently small $\varepsilon > 0$, there is \widehat{A} close to A such that $G(\widehat{A}, \varepsilon) = (\mathbf{c}, \mathbf{d})$. That is, perturbing the (u, v) entry of A to be nonzero, the rest of the nonzero entries of A can be adjusted so that A and $A(\{r, s\})$ have the same characteristic polynomial as before, thus the same eigenvalues as before. \square

Example 5.4 Suppose that we want to find a real symmetric matrix A whose graph is the tree T in Figure 6, such that its eigenvalues are 1, 2, 4, 6, 9, 10 and 12 and the eigenvalues of $A(\{3, 6\})$ are 3, 5, 7, 8 and 11. Note that there is one τ -pairing. In this method, we need to delete an edge on the path from 3 to 6 and add the edge $\{3, 6\}$ and solve the problem

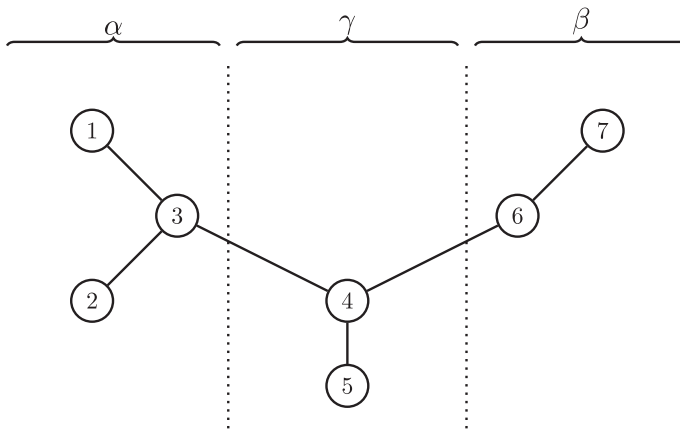


Figure 6. Tree T on 7 vertices where vertices 3 and 6 are not adjacent.

for this tree, but we let $a_{3,6} = 0$. So we have two choices: case I: edge $\{3, 4\}$ and case II: edge $\{4, 6\}$. In either cases $|V_3|, |V_6| > 1$. So, Theorem 5.3 guarantees the existence of such matrix A .

Case 1 $(r, s) = (3, 4)$.

Let T' be the tree obtained from T by deleting the edge $\{3, 4\}$ and inserting the edge $\{3, 6\}$.

Solve the problem for T' with 0 on $(3, 6)$ entry of A . Then, use the Jacobian method to perturb $a_{3,4}$ to $\varepsilon = 0.1$ and adjust other entries to get the following matrix:

$$A \simeq \begin{bmatrix} 3 & 0 & 1.732 & 0 & 0 & 0 & 0 \\ 0 & 7 & 3 & 0 & 0 & 0 & 0 \\ 1.732 & 3 & 4.003 & \mathbf{0.1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0.1} & 6.939 & 1.434 & 4.062 & 0 \\ 0 & 0 & 0 & 1.434 & 6.061 & 0 & 0 \\ 0 & 0 & 0 & 4.062 & 0 & 5.997 & 1.581 \\ 0 & 0 & 0 & 0 & 0 & 1.581 & 11 \end{bmatrix}.$$

Case 2 $(r, s) = (4, 6)$.

Let T' be the tree obtained from T by deleting the edge $\{4, 6\}$ and inserting the edge $\{3, 6\}$.

Solve the problem for T' with 0 on the $(3, 6)$ entry of A . Then use the Jacobian method to perturb $a_{4,6}$ to $\varepsilon = 0.1$ and adjust other entries to get the following matrix:

$$A \simeq \begin{bmatrix} 7 & 0 & 2.372 & 0 & 0 & 0 & 0 \\ 0 & 11 & 1.909 & 0 & 0 & 0 & 0 \\ 2.372 & 1.909 & 6.004 & 3.119 & 0 & 0 & 0 \\ 0 & 0 & 3.119 & 3.960 & 0.999 & \mathbf{0.1} & 0 \\ 0 & 0 & 0 & 0.999 & 4.040 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0.1} & 0 & 3.996 & 3.464 \\ 0 & 0 & 0 & 0 & 0 & 3.464 & 8 \end{bmatrix}.$$

It is easy to check that in both cases the graph of A is T and the eigenvalues of A and $A(\{3, 4\})$ are as desired. Note that the approximations are because of machine error, approximations in finding the roots of the polynomials during the algorithm and also the error in the Newton's method we used to find the roots of the systems of multivariable polynomial equations. Newton's method uses 10 iterations to find above matrices.

Finally, we state an analogue of Theorem 4.7 for the case of connected graphs.

THEOREM 5.5 *Let G be a connected graph on n vertices $1, 2, \dots, n$ where r and s are not adjacent and let $\lambda_1, \dots, \lambda_n, \tau_1, \dots, \tau_{n-2}$ be real numbers satisfying*

$$\lambda_i < \tau_i < \lambda_{i+2}, \tag{14}$$

$$\tau_i \neq \lambda_{i+1}, \tag{15}$$

for all $i = 1 \dots, n - 2$. Furthermore, assume that k τ -pairings occur. If G has a spanning tree T such that $|\alpha|, |\beta| \geq k - |\gamma|$, then there is a real symmetric matrix $A = [a_{ij}]$ with graph G and eigenvalues $\lambda_1, \dots, \lambda_n$ such that eigenvalues of $A(\{r, s\})$ are $\tau_1, \dots, \tau_{n-2}$.

Proof Let T be a spanning tree of G such that $|\alpha|, |\beta| \geq k - |\gamma|$. By Theorem 5.3, there is a matrix A with desired spectrum and subspectrum whose graph is T . The rest of the proof exactly follows as that of Theorem 4.7. \square

6. Diagonal perturbations

Deleting the i -th row and column of a matrix is closely related to perturbing the i -th diagonal entry of the matrix. In this section, the same ideas and techniques of the previous sections are used to study the spectra of a matrix and a diagonal perturbation of the matrix. We begin by studying the case that one diagonal entry is perturbed.

LEMMA 6.1 *Let*

$$\begin{aligned} g(x) &= (x - c_1)(x - c_2) \cdots (x - c_n), \\ h(x) &= (x - d_1)(x - d_2) \cdots (x - d_n), \end{aligned}$$

and

$$f(x) = g(x) - h(x),$$

where $d_1 < c_1 < d_2 < c_2 < \cdots < d_n < c_n$. Then, f has exactly $n - 1$ real roots e_1, \dots, e_{n-1} and they satisfy the inequalities $d_1 < e_1 < d_2 < \cdots < e_{n-1} < d_n$.

Proof First note that the degree of f is $n - 1$, since the coefficient of x^{n-1} in f is the positive quantity $\sum_{i=1}^n c_i - d_i > 0$. Next, note that $g(d_i)$ and $g(d_{i+1})$ have opposite signs for all $i = 1, \dots, n - 1$. Consequently, $f(d_i)$ and $f(d_{i+1})$ have opposite signs for all $i = 1, \dots, n - 1$. Hence, f has a zero between d_i and d_{i+1} for all $i = 1, \dots, n - 1$. That is, f has $n - 1$ roots e_i such that $d_1 < e_1 < d_2 < \cdots < d_{n-1} < e_{n-1} < d_n$. \square

LEMMA 6.2 *Let A be a matrix and let $\hat{A} = A + aE_{ii}$ where E_{ii} is the matrix of the same size as A with its (i, i) entry equal to 1 and all other entries equal to zero. Then,*

$$c_{\hat{A}}(x) = c_A(x) + ac_{A(i)}(x).$$

Proof This follows from the expansion of $\det(xI - \hat{A})$ along the i -th row. \square

The above lemma shows that the eigenvalues of $A(i)$ are roots of $c_{\hat{A}}(x) - c_A(x)$ and this latter polynomial is determined by the eigenvalues of \hat{A} and A . Thus, a solution to the $\lambda - \mu$ structured inverse eigenvalue problem defined by the eigenvalues of A and $A(i)$, is also a solution to the following related diagonal perturbation problem.

THEOREM 6.3 *Let*

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \lambda_n < \mu_n$$

be $2n$ real numbers and i an integer with $1 \leq i \leq n$. Given a tree T , there is an $n \times n$ real symmetric matrix A such that the graph of A is T , A has eigenvalues $\lambda_1, \dots, \lambda_n$ and $A + aE_{ii}$ has eigenvalues μ_1, \dots, μ_n , where $a = \sum_{j=1}^n (\mu_j - \lambda_j) > 0$.

Proof Let

$$f(x) := ((x - \mu_1)(x - \mu_2) \cdots (x - \mu_n)) - ((x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)).$$

By Lemma 6.1, $f(x)$ has exactly $n - 1$ real roots $\gamma_1 < \cdots < \gamma_{n-1}$ which strictly interlace λ_i 's. That is,

$$\lambda_1 < \gamma_1 < \lambda_2 < \gamma_2 < \cdots < \gamma_{n-1} < \lambda_n.$$

So

$$f(x) = \left(\sum_{j=1}^n (\mu_j - \lambda_j) \right) \prod_{i=1}^{n-1} (x - \gamma_i) = a \prod_{i=1}^{n-1} (x - \gamma_i). \tag{16}$$

By Theorem 4.2 of [6], there is a real symmetric matrix A with the Duarte property with respect to vertex i whose graph is T , with eigenvalues $\lambda_1, \dots, \lambda_n$ such that the eigenvalues of $A(i)$ are $\gamma_1, \dots, \gamma_{n-1}$. Let $\widehat{A} = A + aE_{ii}$. By (16), we have

$$\begin{aligned} c_{\widehat{A}}(x) &= c_A(x) + a c_{A(i)}(x) \\ &= (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n) + a(x - \gamma_1)(x - \gamma_2) \cdots (x - \gamma_n) \\ &= (x - \mu_1)(x - \mu_2) \cdots (x - \mu_n), \end{aligned}$$

that is, eigenvalues of $A + aE_{ii}$ are μ_1, \dots, μ_n . □

It is natural to ask if the above matrix A is 'generic' since the matrix obtained in Theorem 4.2 of [6] is. Below, we answer this question in affirmative. We begin with the following technical lemma.

LEMMA 6.4 *Let A be a real symmetric matrix whose graph is a tree T on vertices $1, 2, \dots, n$ and let a be a real positive number. Assume that A has the Duarte property with respect to vertex 1. Let $\mathbf{x} = (x_1, x_2, \dots, x_{2n-1}, y)$ and let $M(\mathbf{x})$ be defined as in Section 4, except for the $(1, 1)$ -entry which is x_1 . Also, let $\widehat{M}(\mathbf{x}) = M(\mathbf{x}) + yE_{11}$. Denote these matrices by M and \widehat{M} for short. Define the function $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ by*

$$f(\mathbf{x}) = (c_0, c_1, \dots, c_{n-1}, d_0, d_1, \dots, d_{n-1}),$$

where c_i and d_i are the nonleading coefficients of the characteristic polynomials of M and \widehat{M} , respectively. Then, Jacobian of f evaluated at $(A, \mathbf{0})$ is nonsingular.

Proof Let $g : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be defined by

$$g(\mathbf{x}) = (c_0, c_1, \dots, c_{n-1}, e_0, e_1, \dots, e_{n-2}),$$

where the c_i are the nonleading coefficients of the characteristic polynomials of M and the e_i are the nonleading coefficients of the characteristic polynomial of $N(\mathbf{x}) = M(\mathbf{x})(1)$ (or N for short). As it is shown by Theorems 3.3 and 4.2 of [6], the Jacobian of g evaluated at

A is nonsingular. Let

$$\text{Jac}(g) \Big|_A = \left[\begin{array}{c} P_{n \times (2n-1)} \\ \hline Q_{(n-1) \times (2n-1)} \end{array} \right],$$

where the rows of P denote the derivatives of the c_i 's evaluated at A and the rows of Q denote the derivatives of the e_i 's evaluated at A .

Observe that $c_{\widehat{M}}(x) = c_M(x) - yc_{M(1)}(x)$. Thus,

$$\text{Jac}(f) \Big|_{(A,a)} = \left[\begin{array}{c|c} P & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline P(;n) - aQ & \begin{matrix} e_0 \\ \vdots \\ e_{n-2} \end{matrix} \\ \hline P_n & 1 \end{array} \right], \tag{17}$$

where P_n is the last row of P and the e_i 's are evaluated at (A, a) . In the matrix in (17), subtract each row of P from the corresponding rows in the second and third block rows and then scale the rows of the second block row by $\frac{1}{a}$. Now the last row is $[0 \cdots 0 \mid 1]$. Subtract appropriate multiples of the last row from each row in the second block row to make all the entries of the last column of that block zero. The resulting matrix

$$\left[\begin{array}{c|c} P & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline Q & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & 1 \end{array} \right],$$

which is row equivalent to $\text{Jac}(f) \Big|_{(A,a)}$ is nonsingular, since $\begin{bmatrix} P \\ Q \end{bmatrix}$ is. □

The proof of the following theorem is similar to the proof of Theorem 4.2 of [6].

THEOREM 6.5 *Let*

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \lambda_n < \mu_n$$

be $2n$ real numbers and i be an integer with $1 \leq i \leq n$. Given a connected graph G , there is an $n \times n$ real symmetric matrix A such that the graph of A is G , A has eigenvalues $\lambda_1, \dots, \lambda_n$ and $A + aE_{ii}$ has eigenvalues μ_1, \dots, μ_n , where $a = \sum_{j=1}^n (\mu_j - \lambda_j) > 0$.

Proof (Sketch of the proof) Since G is a connected graph, it has a spanning tree T . By Theorem 6.3 there is an $n \times n$ real symmetric matrix A with Duarte property with respect to vertex i such that the graph of A is T , A has eigenvalues $\lambda_1, \dots, \lambda_n$ and $A + aE_{ii}$ has eigenvalues μ_1, \dots, μ_n . By Lemma 6.4, the Jacobian of f evaluated at $(A, \mathbf{0})$ is nonsingular. Hence, by Theorem 4.6, for sufficiently small perturbation ϵ of the zero entries of A corresponding to edges in $G \setminus T$ there are adjustments of the diagonal and nonzero off-diagonal entries of A to yield \hat{A} such that $f(\hat{A}, \epsilon) = f(A, \mathbf{0})$. That is, if none of the entries of ϵ are zero and they are sufficiently small. Then the graph of $\hat{A} = G$, A has eigenvalues $\lambda_1, \dots, \lambda_n$ and $A + aE_{ii}$ has eigenvalues μ_1, \dots, μ_n . \square

Now we are ready to study perturbations involving two diagonal entries. Note that one cannot simply perturb one diagonal entry and then perturb another diagonal entry using the above method twice, since the matrix A given by Theorem 6.3 varies for each perturbation.

THEOREM 6.6 *Let $\lambda_1, \dots, \lambda_n$ and τ_1, \dots, τ_n be real numbers such that*

$$\lambda_i < \tau_i < \lambda_{i+2}, \tag{18}$$

$$\tau_i \neq \lambda_{i+1}, \tag{19}$$

for all $i = 1 \dots, n - 2$ and

$$\lambda_{n-1} < \tau_{n-1}, \lambda_n < \tau_n. \tag{20}$$

Assume a graph G satisfies the conditions of Theorems 4.7 or 5.5. Then, there is a real symmetric matrix A and real numbers a_1 and a_2 such that the graph of A is G , the eigenvalues of A are the λ_i 's and the eigenvalues of $A + a_1E_{11} + a_2E_{22}$ are the τ_i 's.

Proof Let T' be a spanning tree of G and T be the forest obtained from T' by deleting the edge $\{u, v\}$ which satisfies the condition in the proof of Theorem 5.3. Note that in the case that 1 and 2 are adjacent in T , then $\{u, v\} = \{1, 2\}$. Call the two obtained connected components T_1 and T_2 , where T_i contains vertex i . By Lemma 2.5 the set of all λ 's and the smallest $n - 2$ τ 's can be partitioned into two sets of sizes at least $2|T_1| - 1$ and $2|T_2| - 1$ such that in each set the τ 's interlace the λ 's. There are two τ 's left, which are the largest τ 's. Assign each of them to one of the sets. By Theorem 6.3, each of these sets can be realized as eigenvalues of a matrix A_i and $A_i + a_iE_{11}$ with graph T_i where A_i has the Duarte property with respect to vertex i for $i = 1, 2$. Let $A = A_1 \oplus A_2$ and $\hat{A} = A + a_1E_{11} + a_2E_{22}$. Let M be the matrix obtained from replacing each diagonal entry of A by $2x_j$, $1 \leq j \leq n$ and by replacing each nonzero off-diagonal entry by x_{n+j} , $1 \leq j \leq n - 2$. Let $N := M + yE_{11} + zE_{22}$.

Define $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ with

$$f(x_1, x_2, \dots, x_{2n-2}, y, z) = (c_0, c_1, \dots, c_{n-1}, d_0, d_1, \dots, d_{n-1}),$$

where c_i and d_i are the nonleading coefficients of M and N , respectively. Define $g : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ with

$$g(x_1, x_2, \dots, x_{2n-2}, y, z) = \left(\frac{\text{tr}M}{2}, \frac{\text{tr}M^2}{4}, \dots, \frac{\text{tr}M^n}{2n}, \frac{\text{tr}N}{2}, \frac{\text{tr}N^2}{4}, \dots, \frac{\text{tr}N^n}{2n} \right).$$

Newton's identities imply that $\text{Jac}(f) \Big|_{(A, a_1, a_2)}$ is nonsingular if and only if $\text{Jac}(g) \Big|_{(A, a_1, a_2)}$ is nonsingular. Note that the Jacobian of g evaluated at (A, a_1, a_2) is:

$$\text{Jac}(g) \Big|_{(A, a_1, a_2)} = \left[\begin{array}{ccc|ccc} I_{i_1 j_1} & \cdots & I_{i_{n-1} j_{n-1}} & I_{11} & \cdots & I_{nn} \\ A_{i_1 j_1} & \cdots & A_{i_{n-1} j_{n-1}} & A_{11} & \cdots & A_{nn} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{i_1 j_1}^{n-1} & \cdots & A_{i_{n-1} j_{n-1}}^{n-1} & A_{11}^{n-1} & \cdots & A_{nn}^{n-1} \\ \hline \widehat{I}_{i_1 j_1} & \cdots & \widehat{I}_{i_{n-1} j_{n-1}} & \widehat{I}_{11} & \cdots & \widehat{I}_{nn} \\ \widehat{A}_{i_1 j_1} & \cdots & \widehat{A}_{i_{n-1} j_{n-1}} & \widehat{A}_{11} & \cdots & \widehat{A}_{nn} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \widehat{A}_{i_1 j_1}^{n-1} & \cdots & \widehat{A}_{i_{n-1} j_{n-1}}^{n-1} & \widehat{A}_{11}^{n-1} & \cdots & \widehat{A}_{nn}^{n-1} \end{array} \right]$$

Reordering the rows and the columns of the above matrix we can write it as $[\text{Jac}_\alpha \mid \text{Jac}_\beta]$, where

$$\text{Jac}_\alpha = \left[\begin{array}{ccc|ccc|c} I[\alpha]_{11} & \cdots & I[\alpha]_{kk} & I[\alpha]_{i_1 j_1} & \cdots & I[\alpha]_{i_{k-1} j_{k-1}} & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A[\alpha]_{11}^{k-1} & \cdots & A[\alpha]_{kk}^{k-1} & A[\alpha]_{i_1 j_1}^{k-1} & \cdots & A[\alpha]_{i_{k-1} j_{k-1}}^{k-1} & 0 \\ \hline A[\alpha]_{11}^k & \cdots & A[\alpha]_{kk}^k & A[\alpha]_{i_1 j_1}^k & \cdots & A[\alpha]_{i_{k-1} j_{k-1}}^k & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A[\alpha]_{11}^{n-1} & \cdots & A[\alpha]_{kk}^{n-1} & A[\alpha]_{i_1 j_1}^{n-1} & \cdots & A[\alpha]_{i_{k-1} j_{k-1}}^{n-1} & 0 \\ \hline I[\alpha]_{11} & \cdots & I[\alpha]_{kk} & I[\alpha]_{i_1 j_1} & \cdots & I[\alpha]_{i_{k-1} j_{k-1}} & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \widehat{A}[\alpha]_{11}^{k-1} & \cdots & \widehat{A}[\alpha]_{kk}^{k-1} & \widehat{A}[\alpha]_{i_1 j_1}^{k-1} & \cdots & \widehat{A}[\alpha]_{i_{k-1} j_{k-1}}^{k-1} & * \\ \hline \widehat{A}[\alpha]_{11}^k & \cdots & \widehat{A}[\alpha]_{kk}^k & \widehat{A}[\alpha]_{i_1 j_1}^k & \cdots & \widehat{A}[\alpha]_{i_{k-1} j_{k-1}}^k & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \widehat{A}[\alpha]_{11}^{n-1} & \cdots & \widehat{A}[\alpha]_{kk}^{n-1} & \widehat{A}[\alpha]_{i_1 j_1}^{n-1} & \cdots & \widehat{A}[\alpha]_{i_{k-1} j_{k-1}}^{n-1} & * \end{array} \right],$$

and

$$\text{Jac}_\beta = \begin{bmatrix} I[\beta]_{kk} & \cdots & I[\beta]_{nn} & I[\beta]_{i_k j_k} & \cdots & I[\beta]_{i_{n-1} j_{n-1}} & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A[\beta]_{kk}^{k-1} & \cdots & A[\beta]_{nn}^{k-1} & A[\beta]_{i_k j_k}^{k-1} & \cdots & A[\beta]_{i_{n-1} j_{n-1}}^{k-1} & 0 \\ \hline A[\beta]_{kk}^k & \cdots & A[\beta]_{nn}^k & A[\beta]_{i_k j_k}^k & \cdots & A[\beta]_{i_{n-1} j_{n-1}}^k & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A[\beta]_{kk}^{n-1} & \cdots & A[\beta]_{nn}^{n-1} & A[\beta]_{i_k j_k}^{n-1} & \cdots & A[\beta]_{i_{n-1} j_{n-1}}^{n-1} & 0 \\ \hline I[\beta]_{kk} & \cdots & I[\beta]_{nn} & I[\beta]_{i_k j_k} & \cdots & I[\beta]_{i_{n-1} j_{n-1}} & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \widehat{A}[\beta]_{kk}^{k-1} & \cdots & \widehat{A}[\beta]_{nn}^{k-1} & \widehat{A}[\beta]_{i_k j_k}^{k-1} & \cdots & \widehat{A}[\beta]_{i_{n-1} j_{n-1}}^{k-1} & * \\ \hline \widehat{A}[\beta]_{kk}^k & \cdots & \widehat{A}[\beta]_{nn}^k & \widehat{A}[\beta]_{i_k j_k}^k & \cdots & \widehat{A}[\beta]_{i_{n-1} j_{n-1}}^k & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \widehat{A}[\beta]_{kk}^{n-1} & \cdots & \widehat{A}[\beta]_{nn}^{n-1} & \widehat{A}[\beta]_{i_k j_k}^{n-1} & \cdots & \widehat{A}[\beta]_{i_{n-1} j_{n-1}}^{n-1} & * \end{bmatrix},$$

where $\widehat{A} := A + a_1 E_{11} + a_2 E_{22}$. In Jac_α , the first two block columns represent derivatives with respect to variables in A_1 , while the third block column (the last column) represents derivatives with respect to y and in Jac_β the first two block columns represent derivatives with respect to variables in A_2 and the third block column (the last column) represents derivatives with respect to z . Furthermore, the first two block rows represent derivatives of $\text{tr}M^i$ and the last two block rows represent the derivatives of $\text{tr}N^i$.

Now suppose

$$\mathbf{r}^T \text{Jac}(g) |_{(A, a_1, a_2)} = \mathbf{0}^T, \tag{21}$$

for some vector $\mathbf{r}^T = (s^T, t^T)$, where $s^T = (s_1, \dots, s_n)$ and $t^T = (t_1, \dots, t_n)$. Let $s(x) = \sum_{i=1}^n s_i x^i$ and $t(x) = \sum_{i=1}^n t_i x^i$. Then, (21) holds if and only if $(s(A) + t(\widehat{A})) \circ A = O$ and $(s(A) + t(\widehat{A})) \circ I = O$, which is equivalent to having

$$(s(A_1) + t(\widehat{A}_1)) \circ A_1 = O, (s(A_1) + t(\widehat{A}_1)) \circ I = O, \tag{22}$$

$$\text{and } (s(A_2) + t(\widehat{A}_2)) \circ A_2 = O, (s(A_2) + t(\widehat{A}_2)) \circ I = O, \tag{23}$$

where I denotes the identity matrix of appropriate size in each case. Let $C_{A_i}(x)$ denote the characteristic polynomial of A_i for $i = 1, 2$. Note that by Cayley–Hamilton theorem [8] $C_{A_i}(A_i) = O$. For $i = 1, 2$ let $s_i(x)$ denote the remainder of division of $s(x)$ by the characteristic polynomial of A_i and t_i denote the remainder of division of $t(x)$ by the characteristic polynomial of \widehat{A}_i . Then, (22) and (23) hold if and only if

$$(s_1(A_1) + t_1(\widehat{A}_1)) \circ A_1 = O, (s_1(A_1) + t_1(\widehat{A}_1)) \circ I = O, \tag{24}$$

$$\text{and } (s_2(A_2) + t_2(\widehat{A}_2)) \circ A_2 = O, (s_2(A_2) + t_2(\widehat{A}_2)) \circ I = O. \tag{25}$$

By Theorem 3.3 of [6], we have $s_i(x) = 0$ and $t_i(x) = 0$ for $i = 1, 2$. So, the characteristic polynomials of A_1 and A_2 divide $s(x)$ and the characteristic polynomials of \widehat{A}_1 and \widehat{A}_2 divide $t(x)$. But, since $c_{A_1}(x)$ and $c_{A_2}(x)$ are relatively prime, $c_{A_1}(x)c_{A_2}(x)$ divides $s(x)$. On the other hand, $\deg(c_{A_1}(x)c_{A_2}(x)) = n$ and $\deg(s(x)) = n - 1$. Hence $s(x) = 0$. Similarly, $t(x) = 0$ and consequently $r(x) = 0$. This proves that the rows of the $\text{Jac}(g)$ evaluated at (A, a_1, a_2) are linearly independent and thus $\text{Jac}(f)$ evaluated at (A, a_1, a_2) is nonsingular.

Similar to the proof of Theorem 4.7, let \mathbf{a} be the assignment of the x_j 's corresponding to A , then $g(\mathbf{a}, a_1, a_2, 0, 0, \dots, 0) = (\mathbf{c}, \mathbf{d})$. There is an open neighborhood U of $(\mathbf{a}, a_1, a_2, 0, \dots, 0)$ each of whose elements has no zeros in the same $2n + 1$ entries. By the Implicit Function Theorem 4.6, there is an open neighborhood V of \mathbf{a} and an open neighborhood W of $\mathbf{0}$ such that $V \times W \subseteq U$ and for each $\mathbf{y} \in W$ there is an $\mathbf{x} \in V$ such that $f(\mathbf{x}, \mathbf{y}) = (\mathbf{c}, \mathbf{d})$. Take \mathbf{y} to be a vector in W with no zero entries on the positions corresponding to the edges in G . Then, the (\mathbf{x}, \mathbf{y}) satisfying $f(\mathbf{x}, \mathbf{y}) = (\mathbf{c}, \mathbf{d})$ corresponds to a matrix $\widehat{A} \in S(G)$ such that the λ 's are the eigenvalues of \widehat{A} and the τ 's are the eigenvalues of \widehat{A} . \square

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