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Construction of matrices with a given graph and prescribed interlaced spectral data

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ABSTRACT

A result of Duarte [Linear Algebra Appl. 113 (1989) 173–182] asserts that for real $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_{n-1}$ with

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \mu_{n-1} < \lambda_n,$$

and each tree T on n vertices there exists an $n \times n$, real symmetric matrix A whose graph is T such that A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and the principal submatrix obtained from A by deleting its last row and column has eigenvalues $\mu_1, \ldots, \mu_{n-1}$. This result is extended to connected graphs through the use of the implicit function theorem.

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1. Introduction

Due to their importance in engineering applications, inverse eigenvalue problems have received considerable attention. Many inverse eigenvalue problems reduce to the construction of a matrix with prescribed spectral data. One interesting problem is based on the *Cauchy interlacing inequalities* for symmetric matrices [1]. Let A be an $n \times n$ real matrix with eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$, and B be an $(n-1) \times (n-1)$ principal submatrix of A with eigenvalues $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_{n-1}$. Cauchy interlacing asserts that

$$\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \mu_{n-1} \leqslant \lambda_n. \tag{1}$$

The corresponding inverse eigenvalue problem is that of constructing an $n \times n$, symmetric matrix with a prescribed structure (e.g., tridiagonal, pentadiagonal, Toeplitz, etc) having eigenvalues $\lambda_1, \ldots, \lambda_n$

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and its trailing $(n-1)\times(n-1)$ principal submatrix having eigenvalues $\mu_1,\dots,\mu_{n-1}$, when the only constraints are those in (1). Many results are known for this problem [2,5,6,8,9,12–18]. Specifically, Duarte [4] has shown that for each such set of spectral data, each $i\in\{1,2,\dots,n\}$, and each tree T on n vertices, there is a symmetric matrix A with the same zero pattern of the adjacency matrix of T, except maybe for the diagonal entries, such that A and the principal submatrix obtained from A by deleting row and column i realize the given spectral data, provided that all the inequalities in (1) are strict. We note that Duarte's result is stated and proved for complex hermitian matrices, but his proof carries over for real symmetric matrices. Throughout the remainder of the paper we restrict our attention to real matrices.

A natural question, and one raised by Wayne Barrett at a recent conference, is: does a similar result hold for arbitrary connected graphs G? This paper answers the question in the affirmative by using an approach similar to the one known as the *Jacobian method* used in the study of spectrally arbitrary patterns [7]. The Jacobian method, which depends on the implicit function theorem, asserts that if a symmetric matrix with the appropriate spectral constraints is "sufficiently generic", then one can realize the prescribed spectral data for each superpattern of the matrix. In section 2, we define and establish the basic properties of such a genericness property, which we call the *Duarte-property*. In section 3, we study the polynomial function that maps the entries of a symmetric matrix to the non-leading coefficients of its characteristic polynomial and the non-leading coefficients of the characteristic polynomial of its trailing principal submatrix, and show that the nonsingularity of this map's Jacobian matrix can be expressed as algebraic conditions on the matrix and its trailing principal submatrix. In section 4, we use the Duarte-property and the implicit function theorem to extend Duarte's result from trees to arbitrary connected graphs.

We conclude this introductory section with a basic matrix theoretic result that will be useful later. The first part of the lemma is well-known (see for example [10]).

Lemma 1.1. *Let A be an $m\times m$ matrix, B be an $n\times n$ matrix, and X be an $m\times n$ matrix such that $AX = XB$. Then the following hold:*

(a) *If A and B do not have a common eigenvalue, then $X = O$.*
(b) *If $X \neq O$ and A and B share exactly one common eigenvalue, then each nonzero column of X is a generalized eigenvector of A corresponding to the common eigenvalue.*

Proof. Note that the condition $AX = XB$ implies that $p(A)X = Xp(B)$ holds for each polynomial $p(x)$. Let $p(x) = m_B(x)$ be the minimal polynomial of B. Then $m_B(A)X = Xm_B(B) = O$. Hence

$$(A-\mu_1 I)\cdots(A-\mu_{n-1}I)X = O, \tag{2}$$

where the μ_i's are the eigenvalues of B. If A and B do not share a common eigenvalue, then each $A-\mu_j I$ is invertible, and it follows that $X = O$.

If A and B share exactly one common eigenvalue, say μ, then each matrix $A-\mu_j I$ with $\mu_j \neq \mu$ is invertible and hence by (2), $(A-\mu I)^k X = O$ for some positive integer k. This implies that each nonzero column of X is a generalized eigenvector of A corresponding to the eigenvalue μ. □

2. The Duarte-property

A key to our main result is showing that for each tree T on n vertices, and each collection of $2n-1$ real numbers satisfying

$$\lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{n-1} < \lambda_n, \tag{3}$$

there exists a "sufficiently generic" $A \in S(T)$ such that $\lambda_1,\dots,\lambda_n$ are the eigenvalues of A and $\mu_1,\dots,\mu_{n-1}$ are the eigenvalues of $A(n)$.

To make the term "sufficiently generic" more precise we need the following definitions. Let A be an $n\times n$ symmetric matrix. For $i\in\{1,2,\dots,n\}$, $A(i)$ denotes the principal submatrix obtained from A by deleting its ith row and column. The *graph of A*, denoted $G(A)$, has vertex set $1,2,\dots,n$ and an

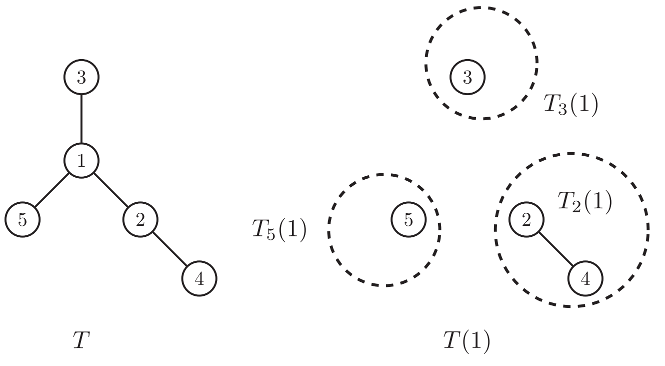


Fig. 1. Graph T [left] and subgraphs obtained by removing vertex 1 [right]

edge joining i and j if and only if $i \neq j$ and $a_{ij} \neq 0$. Thus, $G(A)$ does not depend upon the diagonal entries of A.

Given a graph G with vertex set 1, 2, …, n, $S(G)$ denotes the set of all real, symmetric matrices A whose graph is G; that is, $S(G)$ is the set of all $n \times n$, symmetric matrices $A = [a_{ij}]$ for which $a_{ij} = 0$ if $i \neq j$ and i is not adjacent to j in G, and $a_{ij} \neq 0$ if $i \neq j$ and i is adjacent to j in G.

For a vertex w of T, $T(w)$ denotes the forest obtained from T by deleting the vertex w. If v is a neighbor of w, then $T_v(w)$ denotes the connected component of $T(w)$ having v as a vertex. Note that $T_v(w)$ is necessarily a tree. For $A \in S(T)$, $A(w)$ denotes the principal submatrix of A corresponding to $T(w)$, and $A_v(w)$ denotes the principal submatrix of A corresponding to $T_v(w)$.

For example, Let

$$A = \begin{bmatrix} \mathbf{30} & \mathbf{-2} & \mathbf{-9} & \mathbf{0} & \mathbf{1} \\ \mathbf{-2} & 4 & 0 & -1 & 0 \\ \mathbf{-9} & 0 & -1 & 0 & 0 \\ \mathbf{0} & -1 & 0 & 4 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 2 \end{bmatrix}. \tag{4}$$

Then

$$A(1) = \begin{bmatrix} 4 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

and the graph T of A, $T(1)$ and each of the $T_i(1)$'s are illustrated in Fig. 1.

The matrices related to each $T_i(1)$ are

$$A_2(1) = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}, \quad A_3(1) = \begin{bmatrix} -1 \end{bmatrix}, \quad A_5(1) = \begin{bmatrix} 2 \end{bmatrix}.$$

We now recursively define what it means for a matrix A whose graph is a tree to have the *Duarte-property with respect to a chosen vertex* w:

> If $G(A)$ has just one vertex, then A has the Duarte-property with respect to w. If $G(A)$ has more than one vertex, then A has the Duarte-property with respect to w provided the eigenvalues of $A(w)$ strictly interlace those of A and for each neighbor v of w, $A_v(w)$ has the Duarte-property with respect to the vertex v.

For example, consider the matrix A in (4) and its graph T. Computation shows that the eigenvalues of A are approximately -3.47, 1.98, 2.96, 4.95, 32.58, and eigenvalues of $A(1)$ are $-1, 2, 3, 5$, and they strictly interlace those of A. Furthermore, $T_3(1)$ and $T_5(1)$ are single vertices, so $A_3(1)$ and $A_5(1)$ have the Duarte property. Now we have to verify that $A_2(1)$ has the Duarte property with respect to 2. The eigenvalues of $A_2(1)$ are 3 and 5. The eigenvalue of $(A_2(1))(2)$ is 4, which strictly interlaces 3 and 5. Also $(T_2(1))_4(2)$ is a single vertex, so it has the Duarte property, Hence $A_2(1)$ has the Duarte property. Altogether, this means that A has the Duarte property.

As we shall see in sections 3 and 4, possessing the Duarte-property with respect to a vertex is a sufficient assumption on genericness for our purposes.

For each set of λ's and μ's satisfying (3), Duarte [4] explicitly constructs a matrix $A \in S(T)$ such that the λ's are the eigenvalues of A and the μ's are the eigenvalues of $A(n)$. We now show that Duarte's construction actually yields an A with the Duarte-property with respect to n.

Lemma 2.1. *Let T be a tree with vertices $1, 2, \ldots, n$ with $n \geqslant 2$, w be a chosen vertex and $\lambda_1, \ldots, \lambda_n$, $\mu_1, \ldots, \mu_{n-1}$ be real numbers satisfying (3). Then there exists an $A \in S(T)$ with the Duarte-property with respect to w such that the λ's are the eigenvalues of A and the μ's are the eigenvalues of $A(w)$.*

Proof. The proof is by induction on n. If T has two vertices, then the matrix

$$A = \begin{bmatrix} \mu_1 & \sqrt{(\lambda_2 - \mu_1)(\mu_1 - \lambda_1)} \\ \sqrt{(\lambda_2 - \mu_1)(\mu_1 - \lambda_1)} & \lambda_1 + \lambda_2 - \mu_1 \end{bmatrix}$$

has eigenvalues λ_1 and λ_2, $A(2)$ has eigenvalue μ_1, and A has the Duarte-property with respect to 2. Interchanging the rows of A, and then interchanging the columns, we obtain a matrix with the Duarte-property with respect to 2 and the desired spectral conditions.

Assume $n > 2$ and proceed by induction. Let $v_1, \ldots, v_k$ be the vertices adjacent to w in T, let $g_1(x), g_2(x), \ldots, g_k(x)$ be monic polynomials such that the degree of g_i is the number of vertices of $T_{v_i}(w)$ $(i = 1, 2, \ldots, k)$ and

$$g_1(x)g_2(x)\cdots g_k(x) = \prod_{j=1}^{n-1}(x - \mu_j).$$

As in [4] it can be shown that there exist real numbers a_{ww}, a_{wv_j} $(i = 1, 2, \ldots, k)$ and real, monic polynomials $h_1, \ldots, h_k$ such that

$$\frac{\prod_{i=1}^{n}(x - \lambda_i)}{\prod_{j=1}^{n-1}(x - \mu_j)} = (x - a_{ww}) - \sum_{j=1}^{k} \frac{a_{wv_j}^2 h_j(x)}{g_j(x)}. \tag{5}$$

Also, as in [4], it is possible to show that the roots of h_j are real and strictly interlace those of g_j for each j.

By the induction hypothesis, there exist symmetric matrices $Y_1, \ldots, Y_k$ such that Y_j has graph $T_{v_j}(w)$, Y_j has the Duarte-property with respect to vertex v_j, Y_j's characteristic polynomial is $g_j(x)$ and $Y_j(v_j)$'s characteristic polynomial is $h_j(x)$ $(j = 1, \ldots, k)$.

Let $A = [a_{ij}]$ be the $n \times n$ matrix such that $A_{v_j}(w) = Y_j$, a_{ww}, and $a_{wv_j} = a_{v_jw}$ $(j = 1, 2, \ldots, k)$ are the real numbers defined in (5), and all other entries of A are zero. Then $A \in S(T)$ and, as in Duarte [4], A and $A(w)$ have the desired eigenvalues. Since (3) holds and each Y_j has the Duarte-property with respect to v_j, A has the Duarte-property with respect of w. $\square$

We now show that a matrix with the Duarte-property has a special property, somewhat akin to the strong Arnold property [3]. Given square $n \times n$ matrices R and S we denote their *commutator* by $[R, S]$; that is, $[R, S] = RS - SR$. Given matrices R and S of the same size, $R \circ S$ denotes their Schur (i.e., entrywise) product.

Lemma 2.2. *Let A have the Duarte-property with respect to the vertex w, $G(A)$ be a tree T, and X be a symmetric matrix such that*

(a) $I \circ X = O$,
(b) $A \circ X = O$,
(c) $[A, X](w) = O$,

then $X = O$.

Proof. The proof is by induction on the number of the vertices. Without loss of generality we can take $w = 1$. For $n \leqslant 2$, (a) and (b) imply that $X = O$.

Assume $n \geqslant 3$ and proceed by induction. The matrices A and X have the form

$$A = \left[\begin{array}{c|cccc} a_{11} & b_1^T & b_2^T & \cdots & b_k^T \\ \hline b_1 & A_1 & O & \cdots & O \\ b_2 & O & A_2 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_k & O & O & \cdots & A_k \end{array}\right], \quad X = \left[\begin{array}{c|cccc} 0 & u_1^T & u_2^T & \cdots & u_k^T \\ \hline u_1 & X_{11} & X_{12} & \cdots & X_{1k} \\ u_2 & X_{21} & X_{22} & \cdots & X_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_k & X_{k1} & X_{k1} & \cdots & X_{kk} \end{array}\right],$$

so that each b_i has exactly one nonzero entry and without loss of generality we take this to be in its first position. Thus the A_i's correspond to the $T_v(w)$'s.

The $(2, 2)$-block of $[A, X]$ is

$$b_1 u_1^T + [A_1, X_{11}] - u_1 b_1^T = O.$$

Thus $[A_1, X_{11}] = u_1 b_1^T - b_1 u_1^T$. Since b_1 has just one nonzero entry, the nonzero entries of $u_1 b_1^T - b_1 u_1^T$ lie in its first row or first column. Thus $[A_1, X_{11}](1) = O$. So, A_1 and X_{11} satisfy the induction hypothesis, and thus $X_{11} = O$ and $u_1 b_1^T - b_1 u_1^T = O$. Since the first row of $u_1 b_1^T - b_1 u_1^T$ is a nonzero multiple of u_1^T, we conclude that u_1 is the zero vector. An analogous argument shows that each of $X_{22}, X_{33}, \ldots, X_{kk}, u_2, u_3, \ldots, u_k$ is zero.

Now consider the $(i + 1, j + 1)$-block of $[A, X]$, where $i \neq j$. By (c), $A_i X_{ij} = X_{ij} A_j$. Since A has the Duarte-property with respect to vertex 1, A_i and A_j have no common eigenvalue. So, by part (a) of Lemma 1.1, $X_{ij} = O$. Thus $X = O$. $\square$

3. A polynomial map and its Jacobian matrix

The following will be the setting throughout the remainder of the paper. We fix T to be a tree with vertices $1, 2, \ldots, n$ and edges $e_1 = \{i_1, j_1\}, \ldots, e_{n-1} = \{i_{n-1}, j_{n-1}\}$. Let $x_1, x_2, \ldots, x_{2n-1}$ be $2n - 1$ independent indeterminates, and set

$$x = (x_1, x_2, \ldots, x_{2n-1}).$$

Define $M(x)$ to be the matrix with $2x_i$ in the (i, i) position $(i = 1, 2 \ldots, n)$, x_{n+k} in the (i_k, j_k) and (j_k, i_k) positions $(k = 1, 2, \ldots, n - 1)$, and zeros elsewhere. Set $N(x) = M(x)(n)$; that is, $N(x)$ is the principal submatrix obtained from $M(x)$ by deleting its last row and column. We use M and N to abbreviate $M(x)$ and $N(x)$ when convenient. We note we use $2x_i$ for the (i, i)-position just to make the exposition a bit easier in the proof of the next lemma.

As an example, consider the tree T in Fig. 1. The adjacency matrix of the tree is

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and thus

$$M=\begin{bmatrix} x_1 & x_6 & x_7 & 0 & x_8\\ x_6 & x_2 & 0 & x_9 & 0\\ x_7 & 0 & x_3 & 0 & 0\\ 0 & x_9 & 0 & x_4 & 0\\ x_8 & 0 & 0 & 0 & x_5 \end{bmatrix}, \quad \text{and} \quad N=\begin{bmatrix} x_2 & 0 & x_9 & 0\\ 0 & x_3 & 0 & 0\\ x_9 & 0 & x_4 & 0\\ 0 & 0 & 0 & x_5 \end{bmatrix}.$$

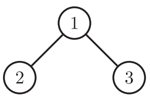
We now define two polynomial maps associated to M and N. Let $g:\mathbb{R}^{2n-1}\to\mathbb{R}^{2n-1}$ be the polynomial map defined by

$$g(x)=(c_0, c_1, \ldots, c_{n-1}, d_0, d_1, \ldots, d_{n-2}),$$

where c_i and d_i are the non-leading coefficients of the characteristic polynomials of M and N, respectively. More precisely, $x^n + c_{n-1}x^{n-1} + \cdots + c_1x^1 + c_0$ and $x^{n-1} + d_{n-1}x^{n-2} + \cdots + d_1x + d_0$ are the characteristic polynomials of M and N, respectively. Let $f:\mathbb{R}^{2n-1}\to\mathbb{R}^{2n-1}$ be the polynomial map defined by

$$f(x)=\left(\frac{\operatorname{tr} M}{2}, \frac{\operatorname{tr} M^2}{4}, \ldots, \frac{\operatorname{tr} M^n}{2n}, \frac{\operatorname{tr} N}{2}, \frac{\operatorname{tr} N^2}{4}, \ldots, \frac{\operatorname{tr} N^{n-1}}{2(n-1)}\right). \tag{6}$$

For example, if we let T be the following graph and construct M as described,



$$M=\begin{bmatrix} 2x_1 & x_4 & x_5\\ x_4 & 2x_2 & 0\\ x_5 & 0 & 2x_3 \end{bmatrix}, \qquad N=\begin{bmatrix} 2x_2 & 0\\ 0 & 2x_3 \end{bmatrix}$$

then $f(x_1, x_2, x_3, x_4, x_5)$ equals

$$\begin{aligned} f(x_1, \ldots, x_5) = \Bigg(& x_1+x_2+x_3, x_1^2+\frac{1}{2}x_4^2+\frac{1}{2}x_5^2+x_2^2+x_3^2,\\ & \frac{4}{3}x_1^3+x_1x_4^2+x_1x_5^2+x_4^2x_2+x_5^2x_3+\frac{4}{3}x_2^3+\frac{4}{3}x_3^3, x_2+x_3, x_2^2+x_3^2\Bigg). \end{aligned}$$

In the next two results, we give a closed formula for the Jacobian matrix of the map f. By Newton's identities, there's an infinitely differentiable, invertible $h:\mathbb{R}^{2n-1}\to\mathbb{R}^{2n-1}$ such that $g\circ h = f$. Thus, the Jacobian matrix of f at a point x is nonsingular if and only if the Jacobian matrix of g at $h(x)$ is nonsingular. We denote a matrix (of size appropriate to the context) with a 1 in position (i, j) and 0s elsewhere by E_{ij}.

Lemma 3.1. *Let (i, j) be a nonzero position of M with corresponding variable x_t. Then*

(a) $\dfrac{\partial}{\partial x_t}\left(\operatorname{tr} M^k\right) = 2kM_{ij}^{k-1}$, *and*

(b) $\dfrac{\partial}{\partial x_t}\left(\operatorname{tr} N^k\right) = \begin{cases} 2kN_{ij}^{k-1} & \text{if neither } i \text{ nor } j \text{ is } n\\ 0 & \text{otherwise.}\end{cases}$

Proof. First, note that if $i\neq j$. Then

$$\frac{\partial}{\partial x_t}M = E_{ij}+E_{ji},$$

and for $i=j$

$$\frac{\partial}{\partial x_t}M = 2E_{ii} = E_{ij} + E_{ji}.$$

Thus, in either case,

$$\begin{aligned}
\frac{\partial}{\partial x_t}\left(\operatorname{tr}(M^k)\right) &= \sum_{\ell=0}^{k-1} \operatorname{tr}\left(M^\ell \cdot \frac{\partial}{\partial x_t}M \cdot M^{k-\ell-1}\right) && \text{(by the chain rule)}\\
&= \sum_{\ell=0}^{k-1} \operatorname{tr}\left(M^{k-1} \cdot \frac{\partial}{\partial t}M\right) && \text{(since } \operatorname{tr}(AB) = \operatorname{tr}(BA) \text{ for any } A \text{ and } B\text{)}\\
&= k \operatorname{tr}\left(M^{k-1}(E_{ij} + E_{ji})\right)\\
&= k\left((M^{k-1})_{ij} + (M^{k-1})_{ji}\right)\\
&= 2k(M^{k-1})_{ij}. && \text{(since } M \text{ is symmetric)}
\end{aligned}$$

A similar argument works for N, provided we note that if i or j equals n then $\frac{\partial}{\partial x_t}N = 0$. $\square$

Given an $(n-1) \times (n-1)$ matrix W, we set

$$\widetilde{W} = \left[\begin{array}{c|c} & 0 \\ W & \vdots \\ & 0 \\ \hline 0 \ \cdots \ 0 & 0 \end{array}\right].$$

Given a matrix $A = [a_{i,j}] \in S(T)$ we denote by $\operatorname{Jac}(f)\Big|_A$ the matrix obtained from $\operatorname{Jac}(f)$ by evaluating at $(x_1, \ldots, x_{2n-1})$ where x_k equals the corresponding entry of A for $k = 1, 2, \ldots, 2n-1$. Lemma 3.1 implies the following.

Corollary 3.2. *Let T be a tree defined as above and $A \in S(T)$. Then*

$$\operatorname{Jac}(f)\Big|_A = \left[\begin{array}{ccc|ccc}
I_{i_1j_1} & \cdots & I_{i_{n-1}j_{n-1}} & I_{11} & \cdots & I_{nn}\\
A_{i_1j_1} & \cdots & A_{i_{n-1}j_{n-1}} & A_{11} & \cdots & A_{nn}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
A^{n-1}_{i_1j_1} & \cdots & A^{n-1}_{i_{n-1}j_{n-1}} & A^{n-1}_{11} & \cdots & A^{n-1}_{nn}\\
\hline
\widetilde{I}_{i_1j_1} & \cdots & \widetilde{I}_{i_{n-1}j_{n-1}} & \widetilde{I}_{11} & \cdots & \widetilde{I}_{nn}\\
\widetilde{B}_{i_1j_1} & \cdots & \widetilde{B}_{i_{n-1}j_{n-1}} & \widetilde{B}_{11} & \cdots & \widetilde{B}_{nn}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
\widetilde{B}^{n-2}_{i_1j_1} & \cdots & \widetilde{B}^{n-2}_{i_{n-1}j_{n-1}} & \widetilde{B}^{n-2}_{11} & \cdots & \widetilde{B}^{n-2}_{nn}
\end{array}\right]$$

The aim now is to show that the above Jacobian matrix is nonsingular whenever A has the Duarte-property with respect to n.

Theorem 3.3. *Let A, B and the function f be defined as above. If A has the Duarte-property with respect to vertex n, then $\mathrm{Jac}(f)\Big|_A$ is nonsingular.*

Proof. Note that $\mathrm{Jac}(f)\Big|_A$ is nonsingular if and only if the only vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2n-1})^T$ such that $\alpha^T \mathrm{Jac}(f)\Big|_A = (0, \ldots, 0)$ is the zero-vector.

Let $\mathrm{Jac}(f)_k$ denote the kth row of $\mathrm{Jac}(f)\Big|_A$. So $\alpha^T \mathrm{Jac}(f)\Big|_A = \sum_{k=1}^{2n-1} \alpha_i \mathrm{Jac}(f)_k$. Thus, for $\ell \leqslant n-1$, the ℓ-th entry in $\alpha^T \mathrm{Jac}(f)\Big|_A$ is the (i_ℓ, j_ℓ)-entry of $\sum_{k=0}^{n-1} \alpha_k A^k + \sum_{k=0}^{n-2} \alpha_{n+k} \widetilde{B}^k$, and for $\ell > n-1$ the ℓ-th entry in $\alpha^T \mathrm{Jac}(f)\Big|_A$ is the $(\ell - n + 1, \ell - n + 1)$-entry of $\sum_{k=0}^{n-1} \alpha_k A^k + \sum_{k=0}^{n-2} \alpha_{n+k} \widetilde{B}^k$. Thus, we have shown that $\alpha^T \mathrm{Jac}(f)\Big|_A$ is the zero vector if and only if the matrix

$$X = \alpha_1 I + \alpha_2 A + \cdots + \alpha_n A^{n-1} + \alpha_{n+1} \widetilde{I} + \alpha_{n+2} \widetilde{B}^1 + \cdots + \alpha_{2n-1} \widetilde{B}^{n-2}$$

satisfies $X \circ A = O$ and $X \circ I = O$.

Let $p(x) = \sum_{i=1}^{n} \alpha_i x^{i-1}$ and $q(x) = \sum_{j=n+1}^{2n-1} \alpha_j x^{j-(n+1)}$. Then $X = p(A) + \widetilde{q(B)}$ and to show that $\mathrm{Jac}(A)$ is nonsingular it suffices to show that $p(x)$ and $q(x)$ are both zero polynomials.

Note that $[A, p(A)] = O$, hence $[A, X] = [A, \widetilde{q(B)}]$. Also, note that since $A(n) = B$, $[A, \widetilde{q(B)}](n) = O$. Thus, $[A, X](n) = O$, and, by Lemma 2.2, we conclude that $X = O$. This implies that $p(A) = -\widetilde{q(B)}$. Let $Y := p(A) = -\widetilde{q(B)}$, then $AY = Ap(A)$. We claim that $Y = O$. Calculations yield:

$$Ap(A) = -A(\widetilde{q(B)}) = -\left[\begin{array}{ccc|c} & & & * \\ & B & & * \\ & & & * \\ \hline * & \cdots & * & * \end{array}\right]\left[\begin{array}{ccc|c} & & & 0 \\ & q(B) & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} & & & 0 \\ & -Bq(B) & & \vdots \\ & & & 0 \\ \hline * & \cdots & * & 0 \end{array}\right],$$

and

$$p(A)A = -\left[\begin{array}{ccc|c} & & & 0 \\ & q(B) & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right]\left[\begin{array}{ccc|c} & & & * \\ & B & & \vdots \\ & & & * \\ \hline * & \cdots & * & * \end{array}\right] = \left[\begin{array}{ccc|c} & & & * \\ & -q(B)B & & \vdots \\ & & & * \\ \hline 0 & \cdots & 0 & 0 \end{array}\right].$$

Since $Ap(A) = p(A)A$, the last row of $Ap(A)$ is zero and the last column of $Ap(A)$ is zero. Thus, $Ap(A) = -\widetilde{q(B)}\widetilde{B} = p(A)\widetilde{B}$. That is, $AY = Y\widetilde{B}$. Hence, by (a) of Lemma 1.1 either $Y = O$, or A and $\widetilde{B}$ have a common eigenvalue. If $Y = O$ we are done. Otherwise, since A and B have no common eigenvalue, A and $\widetilde{B}$ both have an eigenvalue 0 of multiplicity one. Suppose column j of Y is nonzero, and let Y_j denote this column. Note the last entry of Y_j is 0. Since $AY = O$. By (b) of Lemma 1.1, Y_j is a generalized eigenvector of A corresponding to 0. Since A is symmetric, Y_j is an eigenvector of A corresponding to 0. The form of A and the fact that the last entry of Y_j is 0 imply that the vector $Y_j(n)$ is a nonzero eigenvector of

B corresponding to 0. This leads to the contradiction that A and B have a common eigenvalue. Thus $Y = O$.

Since $Y = O$, $p(A) = O$ and $q(B) = O$. Note that $p(x)$ is a polynomial of degree at most $n - 1$. Since A has n distinct eigenvalues, its minimal polynomial has degree n. Thus $p(x)$ is the zero polynomial. Similarly $q(x)$ is the zero polynomial. So $\mathrm{Jac}(f)\Big|_A$ is nonsingular. $\square$

4. Main result

We use the Implicit Function Theorem, a version of which we state below for convenience, to prove our main result (see [11]).

Theorem 4.1. *Let $F : \mathbb{R}^{s+r} \to \mathbb{R}^s$ be a continuously differentiable function on an open subset U of $\mathbb{R}^{s+r}$ defined by*

$$F(x, y) = (F_1(x, y), F_2(x, y), \ldots, F_s(x, y)),$$

where $x = (x_1, \ldots, x_s) \in \mathbb{R}^s$ and $y \in \mathbb{R}^r$. Let (a, b) be an element of U with $a \in \mathbb{R}^s$ and $b \in \mathbb{R}^r$, and c be an element of $\mathbb{R}^s$ such that $F(a, b) = c$. If

$$\left[\frac{\partial F_i}{\partial x_j}\bigg|_{(a,b)} \right]$$

is nonsingular, then there exist an open neighborhood V containing a and an open neighborhood W containing b such that $V \times W \subseteq U$ and for each $y \in W$ there is an $x \in V$ with $F(x, y) = c$

We are now ready to state and prove our main result.

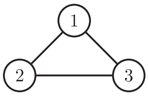
Theorem 4.2. *Let G be a connected graph with vertices $1, 2, \ldots, n$; i a vertex of G, and $\lambda_1, \ldots, \lambda_n$, $\mu_1, \ldots, \mu_{n-1}$ real numbers satisfying (3). Then there is a symmetric matrix $A = \left[a_{ij}\right]$ with graph G and eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $A(i)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$.*

Proof. Without loss of generality $i = n$. Let T be a spanning tree of G. Lemma 2.1 implies that there exists an $A \in S(T)$ such that A has eigenvalues $\lambda_1, \ldots, \lambda_n$, $A(n)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$, and A has the Duarte-property with respect to n. By Theorem 3.3, the Jacobian matrix of the function f defined in (6) evaluated at A is nonsingular. Thus, the Jacobian matrix of the function g defined in section 3 at A is nonsingular.

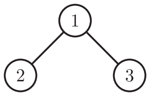
Assume that G has r edges not in T and let $y_1, \ldots, y_r$ be r new variables other than $x_1, \ldots, x_{2n-1}$. We can extend the function $g : \mathbb{R}^{2n-1} \to \mathbb{R}^{2n-1}$ to a function $F : \mathbb{R}^{(2n-1)+r} \to \mathbb{R}^{2n-1}$ by replacing each pair of entries of M (and N) corresponding to an edge of G not in T by one of the y_i's, and let $G(x, y)$ be the vector of nonleading coefficients of the characteristic polynomials of M and N. Let c and d be the vectors of nonleading coefficients of the characteristic polynomials of A and $A(n)$, respectively.

Letting a be the assignment of the x_j's corresponding to A we see that $g(a, 0, 0, \ldots, 0) = (c, d)$. Since each of the first $n - 1$ entries of a is nonzero, there is an open neighborhood U of $(a, 0, \ldots, 0)$ each of whose elements has no zeros in its first $n - 1$ entries. By Theorem 4.1, there is an open neighborhood V of a and an open neighborhood W of $(0, 0, \ldots, 0)$ such that $V \times W \subseteq U$ and for each $y \in W$ there is an $x \in V$ such that $F(x, y) = (c, d)$. Take y to be a vector in W with no zero entries. Then the (x, y) satisfying $F(x, y) = (c, d)$ corresponds to a matrix $\widehat{A} \in S(G)$ such that the λ's are the eigenvalues of $\widehat{A}$ and the μ's are the eigenvalues of $\widehat{A}(n)$. $\square$

Here we give a simple example to illustrate how this method works. Suppose $G = K_3$ and $i = 1$. We want to construct a 3×3 matrix A with prescribed eigenvalues, say -10, 0 and 2 such that the eigenvalues of $A(1)$ are prescribed and interlace those of A, say -1 and 1, and $G(A) = K_3$. First, we choose an spanning tree of G and apply Duarte's method on it to realize the given spectral data.



G



T

The adjacency matrix of T is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Let

$$\hat{A} = \begin{bmatrix} a & d & e \\ d & b & 0 \\ e & 0 & c \end{bmatrix}.$$

Since $A(1)$ is going to be a diagonal matrix with eigenvalues $-1, 1$, we have $b = -1$, $c = 1$. Also, we want the characteristic polynomial of A to satisfy

$$c_A(\lambda) = (\lambda + 10)(\lambda)(\lambda - 2) = \lambda^3 + 8\lambda^2 - 20\lambda,$$

and

$$g(\lambda) = g_1(\lambda)g_2(\lambda) = (\lambda + 1)(\lambda - 1) = \lambda^2 - 1.$$

Then

$$\frac{c_A(\lambda)}{g(\lambda)} = \lambda - (\mathbf{-8}) - \left(\frac{\mathbf{27}}{\mathbf{2}}\frac{1}{\lambda + 1} + \frac{\mathbf{11}}{\mathbf{2}}\frac{1}{\lambda - 1}\right).$$

So, $a = -8$, $d = \frac{27}{2}$, and $e = \frac{11}{2}$. thus

$$A = \begin{bmatrix} -8 & \sqrt{\frac{27}{2}} & \sqrt{\frac{11}{2}} \\ \sqrt{\frac{27}{2}} & -1 & 0 \\ \sqrt{\frac{11}{2}} & 0 & 1 \end{bmatrix}$$

realizes the given spectral data, and has the Duarte property. M and N are described in the previous example and the function f is given. We calculate the Jacobian of f

$$\mathrm{Jac}(f) = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 1 \\ x_4 & x_5 & 2x_1 & 2x_2 & 2x_3 \\ 2x_1x_4 + 2x_2x_4 & 2x_1x_5 + 2x_3x_5 & 4x_1^2 + x_4^2 + x_5^2 & 4x_2^2 + x_4^2 & 4x_3^2 + x_5^2 \\ \hline 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2x_2 & 2x_3 \end{array}\right]$$

By direct calculation $\det(\mathrm{Jac}(f)) = 4\,x_4x_5x_3^2 - 8\,x_4x_5x_3x_2 + 4\,x_5x_4x_2^2$, and

$$\det\left(\mathrm{Jac}(f)\Big|_A\right) = 12\sqrt{132}.$$

So, the implicit function theorem tells us that if we change the zero entries to some small nonzero number, there will be numbers close to the entries of A such that the new matrix constructed with these new numbers realizes the given spectral data. For example let $y = \sqrt{3}/2$, then the matrix

$$B = \begin{bmatrix} -8 & \frac{9+\sqrt{11}}{2\sqrt{2}} & \frac{\sqrt{66}-3\sqrt{6}}{4} \\ \frac{9+\sqrt{11}}{2\sqrt{2}} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{66}-3\sqrt{6}}{4} & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

has the eigenvalues -10, 0 and 2 and

$$B(1) = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

has eigenvalues -1 and 1.

Given $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ it is easy to find $\mu_1, \mu_2, \ldots, \mu_{n-1}$ such that (3) holds and hence Theorem 4.2 immediately implies the following corollary. This extends Theorem 2 of [4] from trees to connected graphs.

Corollary 4.3. *Let G be a connected graph on n vertices and $\lambda_1, \lambda_2, \ldots, \lambda_n$ distinct real numbers. Then there exists a matrix $A \in S(G)$ whose spectrum is $\lambda_1, \ldots, \lambda_n$.*

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