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Permanents were introduced by Cauchy and Binet in early 19^{th} century. Permanents now have applications in combinatorics, graph theory, probability theory, and computational complexity theory.

The permanent of the $n \times n$ matrix $A = \left[a_{ij}\right]$ is defined to be the sum of all diagonal products of A, that is:

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where S_n is the symmetric group of order n.

The term rank of A, denoted termrank(A), is the largest number of nonzero entries of A with no two in the same row or column. The permanent rank of a matrix has been defined by Yang Yu in his 1999 Ph.D. dissertation. The permanent rank of a matrix A, denoted by perrank(A), is defined to be the size of a largest square sub-matrix of A with nonzero permanent.

In this thesis we study some properties of the permanents and also permanent ranks of the matrices, and related problems. In particular we characterize matrices with small permanent ranks in order to study the following conjecture relating the perrank and the termrank:

Conjecture: For any matrix A,

$$\operatorname{perrank}(A) \geq \left\lceil \frac{\operatorname{termrank}(A)}{2} \right\rceil,$$

and for even termrank the equality holds if and only if up to equivalence and reduction $A = \bigoplus \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

ON THE PERMANENT RANK OF MATRICES

by

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To my parents, Taybeh and Hassan

Contents

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Chapter 1

Preface

This thesis concerns the matrix function known as the permanent and a concept known as permanent rank. The term "permanent" seems to have originated in Cauchy's memoir of 1812 [8], where he developed the theory of determinants as a special type of alternating symmetric function that he distinguished from the ordinary symmetric functions [13]. The permanent of the n by n matrix $A = \left[a_{ij}\right]$ is defined to be [13] the sum of all diagonal products of A, that is:

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the S_n is the symmetric group of order n.

The permanent appears in many places in mathematics. As H.J. Ryser mentions in *Combinatorial Mathematics* [16], the permanent "appears repeatedly in the literature of combinatorics, in connection with certain enumeration and extremal problems." For example it can represent the number of the *cycle covers* of a directed graph, or the number of *perfect matchings* in a bipartite graph. These topics will be discussed in Chapter 2. There are many open problems related to the computing of perfect matchings [14].

Note that the determinant of the n by n matrix $A = [a_{ij}]$ is defined as the sum of the all *signed* diagonal products of a matrix, namely,

$$\det(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} \operatorname{sgn}(\sigma) a_{i\sigma(i)}.$$

Using Gaussian elimination, the determinant of an n by n matrix can be calculated in $O(n^3)$ steps. In spite of its similarity to the determinant, the computation of the permanent *seems to be* more computationally difficult to calculate. Indeed there is no known polynomial-time algorithm for calculating the permanent of $n \times n$ $(0,1)$-matrices [15]. Furthermore, L. Valiant [21] proved that calculating the permanent has at least the difficulty of any problem in the class of counting problems known as #P problems. Thus the class of #P problems includes counting problems such as determining the number of subsets of a given set of integers whose elements sums to 0, or the number of Hamiltonian cycles in a given weighted graph which have cost less than a fixed constant k.

Knowing a little about permanents we are interested to study the *permanent rank* of matrices. Recall that the rank of a matrix A is defined to be the maximum number of independent rows or columns of the matrix [22], which is equivalent to the order of the largest square sub-matrix of A with nonzero determinant [9]. Analogously, Yang Yu [24] in his PhD thesis defines the permanent rank of a matrix A, denoted by $\operatorname{perrank}(A)$, to be the order of a largest square sub-matrix of A with nonzero permanent. Despite the fact that no significant work has been done on the permanent rank since 1999, it has some significant properties and applications, and thus there is interest in better understanding the permanent rank.

Motivated by the Alon-Jaeger-Tarsi (AJT) conjecture, mentioned below, Yu proposed another conjecture which states that for every $n \times n$ nonsingular matrix A over a given but arbitrary field, $\operatorname{perrank}\left(\begin{array}{c|c} A & A \end{array} \right) = n$. As noted by Yu [23] this conjecture implies the AJT Conjecture [2]:

> If A is an n times n nonsingular matrix A over a field $\mathbb{F}$ with at least four elements there is a nowhere-zero vector x, such that Ax is also nowhere-zero. Here *nowhere-zero* means a vector such that none of the components are zero.

Having done some research on the AJT conjecture in my undergraduate studies, I decided to study the permanent rank of matrices and related problems for my master's thesis.

Alon and Tarsi [2] proved the AJT Conjecture is true for any any field which is not of a prime order, that is for $|\mathbb{F}| = p^q$, for some prime p and integer $q \geq 2$. The problem is still open for $|\mathbb{F}| = 5$. The AJT Conjecture is connected to different areas in mathematics. For example, Alon and Tarsi's proof uses a combinatorial analog of Hilbert's Nullstellensatz for the existence of solutions to polynomial equations, which has been a useful tool in other areas of mathematics [3]. Also, when $\mathbb{F}$ has nonzero characteristic, the AJT Conjecture is known to be intimately related to the problem of covering $\mathbb{F}^n$ with $2n$ hyperplanes [10, 17].

Not surprisingly, there is another conjecture which is implied by AJT Conjecture. It is known as the 3-*flow conjecture*, which was purposed by W.T. Tutte in [20]. A graph is k-edge-connected if it remains connected whenever any set of fewer than k edges is removed. An m-flow is a weighting of edges of the graph over $\mathbb{Z}_m$, the integers modulo m, such that the sum of input flows is equal to the sum of output flows at each node. An m-flow is a *nowhere-zero flow* if none of the edges has zero weight. The conjecture asserts that every 4-*edge-connected* graph has a nowhere-zero 3-*flow* [19]. It should be mentioned that Jaeger was able to prove that every 4-edge-connected graph has a 4-flow, but very little is known about 3-flows. So, he actually made another conjecture which says there is an integer k such that every k-edge-connected graph has a nowhere-zero 3-flow. This is known as the weak 3-flow conjecture. Recently, C. Thomassen [18, 19] has proven the conjecture is true, with $k = 8$. This problem has applications in coloring planar graphs (by duality) and graph decompositions [18].

Chapter 2

On the permanent rank of matrices

In this chapter the permanent rank of matrices will be studied, and a characterization of matrices of small permanent rank will be presented. Then we will propose a conjecture relating the permanent rank of matrices to their term rank, which suggests that the permanent rank of a matrix, unlike its rank, is more related to the zero pattern of the matrix, rather than the individual entires.

2.1 Motivation

In his 1998 PhD thesis Yang Yu [24] defined the perrank of a matrix $A \in M_n(\mathbb{F})$ to be the size of a largest sub-matrix of A with nonzero permanent. He proves two important results for the perrank:

Proposition 2.1.1. *Let A, B be $n \times n$ matrices over the field $\mathbb{F}$. Then*

a) $\operatorname{perrank}(A) \geq \dfrac{\operatorname{rank}(A)}{2}$. *This is Theorem 2.1.2.*

b) *If A, B have no zero entries, and* $\operatorname{char}(\mathbb{F}) = 0$, *then* $\left[\begin{array}{c|c} A & B \end{array}\right]$ *has* $\operatorname{perrank} n$. *This is an immediate result of Lemma 7.7 of [24].*

Notation: The are two ways to denote a sub-matrix of a matrix $A_{m\times n}$. Let $\alpha \subset \{1, \dots, m\}$

and $\beta \subset \{1, \ldots, n\}$. The sub-matrix of A obtained from rows of A indexed by α and columns of A indexed by β is denoted by $A[\alpha;\beta]$. The sub-matrix of A obtained by omitting those rows of A indexed by α and columns of A indexed by β is denoted by $A(\alpha;\beta)$. If we want to keep all the rows of A and omit just the columns of A which are indexed by β we use the notation $A(-;\beta)$. Similarly, when no columns of A are omitted the notation $A(\alpha;-)$ is used. Similarly, $A[\alpha;\ ]$ is used to denote a sub-matrix of A keeping only those rows indexed by α and all the columns, and $A[\ ;\beta]$ is the sub-matrix keeping all the rows and only the columns indexed by β.

Yu uses some lemmas and provides a nice proof for part (a) of the Proposition 2.1.1 [23]. We also provide another proof which is more direct.

Theorem 2.1.2. *Let $A_{m\times n}$ be a rank r matrix. Then*

$$\operatorname{perrank}(A) \geq \frac{\operatorname{rank}(A)}{2}.$$

Proof: Without loss of generality assume that $m \leq n$, and $\operatorname{rank}(A) = m$. Suppose $\operatorname{perrank}(A) = k$, and B is a $k \times k$ sub-matrix of A with nonzero permanent. Then by permuting rows and columns we may take A to have the form

$$A = \left[\begin{array}{c|c} B & X \\ \hline Y & Z \end{array}\right].$$

Since $\operatorname{perrank}(A) = k$, the permanent of each $(k+1)\times(k+1)$ sub-matrix of A is zero. In particular $\operatorname{per}\left(\begin{array}{c|c} B & X[\ ;j] \\ \hline Y[i;\ ] & Z_{i,j} \end{array}\right) = 0$, for all i, j. Thus

$$Z_{i,j} = \frac{-\operatorname{per}\left(\begin{array}{c|c} B & X[\ ;j] \\ \hline Y[i;\ ] & 0 \end{array}\right)}{\operatorname{per}(B)}.$$

Define $\operatorname{peradj}(B)$ to be the $k \times k$ matrix whose $(i,j)^{th}$ entry is $\operatorname{per}(B(i,j))$. Define $\hat{B}$ as $\frac{\operatorname{peradj}(B)}{\operatorname{per}(B)}$. Hence the $(i,j)^{th}$ entry of $-Y\hat{B}X$ is

$$\frac{-\left(Y\operatorname{peradj}(B)X\right)_{i,j}}{\operatorname{per}(B)} = \frac{-1}{\operatorname{per}(B)}\left[Y[i;\ ]\right]\left[\operatorname{per}(B(r,s))\right]\left[X[\ ;j]\right] = Z_{i,j}.$$

Therefore

$$A = \left[\begin{array}{c|c} B & X \\ \hline & \\ Y & -Y\hat{B}X \\ & \end{array}\right].$$

Note that Y is an $(m-k) \times k$ matrix. If $m-k > k$, then the rows of Y, and indeed the rows of $\left[\begin{array}{c|c} Y & -Y\hat{B}X \end{array}\right]$ are linearly dependent. But this contradicts that $\text{rank}(A) = m$. So, $m - k \leq k$, that is, $k \geq m/2$. $\square$

Remark 2.1.3. *If A is invertible and* $\text{perrank}(A) = n/2$*, the proof of Theorem 2.1.2 shows that every $k \times k$ sub-matrix D of A with nonzero permanent is contained in a sub-matrix of A with permanent rank $n/2$.*

Yu's work on the perrank was motivated by the following conjecture due to Alon, Jaeger and Tarsi [2]:

Conjecture 2.1.4 (AJT Conjecture)**.** *If $\mathbb{F}$ is a field with $|\mathbb{F}| \geq 4$, A is a nonsingular matrix over $\mathbb{F}$, then there is a vector x such that both x and Ax have only nonzero entries.*

Alon and Tarsi proved the conjecture is true for non-prime fields [2]. But even the case $|\mathbb{F}| = 5$ is still open. It is easy to see why it is true over the field of real numbers $\mathbb{R}$. Take a nowhere-zero vector x, such that Ax has minimum number of the zeros. If Ax has no zeros then we are done. Otherwise, there exist a j such that $(Ax)_j = 0$. A is invertible, so there is an i such that $a_{ij} \neq 0$. Now, modify the choice of x by perturbing the i^{th} entry by adding ε. Let e_i be the unit vector with a 1 on the i^{th} positions and 0's everywhere else. There exist an ε such that $x + \varepsilon e_i$ remains nowhere-zero, and in addition, none of the nonzero entries of Ax will turn to zero. Furthermore, since $a_{ij} \neq 0$, $(A(x + \varepsilon e_i))_j$ is also nonzero. That is, $(A(x + \varepsilon e_i))$ has fewer zero entries than Ax. But x was chosen such that Ax has minimum number of zeros. Hence, Ax cannot have zero entries.

Note that we use the assumption of A being invertible only to guarantee there is a nonzero element in each row of A in the above proof. Hence, the conjecture over a *complete* field, like the real numbers, is true even if the condition of A being invertible is relaxed to A has no zero rows.

Furthermore, in the above proof for the field of real numbers, we have heavily used the properties of the real numbers (such as completeness). There is another easy proof, which uses basic properties of vector spaces, that works for any infinite field. It is easily seen that the union of two subspaces of a vector space is a subspace if and only if one of them is a subset of the other one [6].

Proposition 2.1.5. *Let K, L be subspaces of a vector space V. Then $K \cup L$ is a subspace if and only if $K \subseteq L$ or $L \subseteq K$.*

Proof: Suppose neither K is a subset of L nor L is a subset of K. Then there are vectors $x \in K \setminus L$ and $y \in L \setminus K$. But then $x + y$ is neither in K nor in L; that is $x + y \notin K \cup L$. Therefore $K \cup L$ is not closed under addition, and hence is not a subspace. □

Furthermore, one can argue that a vector space over an infinite field, cannot be written as a finite union of its *proper* subspaces [12].

Proposition 2.1.6. *Let V be a vector space over an infinite field $\mathbb{F}$. Then V is not a union of finitely many proper subspaces of itself.*

Proof: Assume $V = \bigcup_{i=1}^{n} V_i$, for some $n \geq 2$, where V_i are proper subspaces of V, and n is minimal, that is, $V_i \nsubseteq \bigcup_{j \neq i} V_j$, for all i. Then there exist nonzero vectors $u \notin V_n$ and $v \in V_n \setminus \bigcup_{i \neq n} V_i$. Let $S = \{v + tu : t \in \mathbb{F}\}$. Then $S \subseteq V$ is an infinite set. Since, $S \subseteq V = \bigcup_{i=1}^{n} V_i$, one of the V_i must contain infinitely many elements of S. Since $u \notin V_n$, V_n does not contain any element of S other than v. Suppose V_j $(j \neq n)$ contains infinitely many elements of S. Then, there are $t_1, t_2 \in \mathbb{F}$ $(t_1 \neq t_2)$, such that $v + t_1 u$, $v + t_2 u \in V_j$. Thus $v = (t_2 - t_1)^{-1}(t_2(v + t_1 u) - t_1(v + t_2 u)) \in V_j$. That contradicts the choice of v. Hence V cannot be the union of finitely many subspaces of itself. □

We can use this fact to prove the AJT Conjecture for infinite fields.

Theorem 2.1.7. *If $\mathbb{F}$ is an infinite field, and $A_{n \times n}$ is a nonsingular matrix over $\mathbb{F}$, then there is a vector x such that both x and Ax have only nonzero entries.*

Proof: Let $V_i = \{v \in \mathbb{F}^n : v_i = 0\}$, and $W_i = \{v \in \mathbb{F}^n : (Av)_i = 0\}$, for $i \in \{1, \ldots, n\}$. Clearly V_i and W_i are subspaces of $\mathbb{F}^n$. A vector x and Ax both have no zero entries if and only if $x \notin V_i$ and $x \notin W_i$, for all i; or equivalently if and only if $x \notin (\bigcup_{i=1}^n V_i) \cup (\bigcup_{i=1}^n W_i)$. But $\mathbb{F}$ is an infinite field, and then by Proposition 2.1.6 such x exists, since

$$\mathbb{F}^n \setminus \left((\bigcup_{i=1}^n V_i) \cup (\bigcup_{i=1}^n W_i) \right) \neq \emptyset. \quad \square$$

Using this point of view, one can translate the AJT Conjecture to a problem of finding out when an n-dimensional vector space is not covered by the hyperplanes $V_i = \{v \in \mathbb{F}_n : v_i = 0\}$, and $W_i = \{v\mathbb{F}_n : (Av)_i = 0\}$, where A is an invertible matrix. In Lemma 19 of [17], Szegedy gives the formal statement of this relation and provides a proof for it. Also, the authors of [1] give a characterization for these spaces, when the size of the matrix is small with respect to the size of the field.

Jeff Kahn, Yang Yu's PhD supervisor, initiated the study of perrank and was led to the following conjecture:

Conjecture 2.1.8. *For any $n \times n$ nonsingular matrix A over any field,*

$$\operatorname{perrank}(\left[\begin{array}{c|c} A & A \end{array}\right]) = n.$$

Conjecture 2.1.8 implies Conjecture 2.1.4, using the polynomial method of Alon and Tarsi [2]. In that method, Alon and Tarsi considered a polynomial which is the product of the entries of x and Ax for $A = [a_{ij}]$. In other words

$$P(x_1, x_2, \ldots, x_n) = \prod_{i=1}^n x_i \prod_{i=1}^n \left(\sum_{j=1}^n a_{ij} x_j \right).$$

The only step that is needed to complete the proof is that this polynomial is not constantly zero. They prove this using a direct consequence of the combinatorial version of the Nullstellensatz [3], which says for this polynomial it is enough to show that at least one of the coefficients is not zero:

Theorem 2.1.9. *Let $\mathbb{F}$ be an arbitrary field, and let $f = f(x_1, \ldots, x_n)$ be a polynomial in $\mathbb{F}[x_1, \ldots, x_n]$. Suppose the degree of f is $\sum_{i=1}^n t_i$, where each t_i is a nonnegative integer, and* *suppose the coefficient of* $\prod_{i=1}^{n} x^{t_i}$ *in* f *is nonzero. Then, if* $S_1, \ldots, S_n$ *are subsets of* $\mathbb{F}$ *with* $|S_i| > t_i$*, there are* $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ *so that* $f(s_1, \ldots, s_n) \neq 0$.

Alon and Tarsi use the fact that the coefficient of each $\prod x^{\alpha}, \alpha = (\alpha_1, \ldots, \alpha_n)$, is $\mathrm{per}(A_\alpha)$, where the columns of A_α are α_i copies of the i^{th} column of A. Then the permanent of A is also one of the coefficients, particularly it is the coefficient of $x_1 \ldots x_n$. On the other hand, every A_α is an $n \times n$ sub-matrix of $\left[\begin{array}{c|c|c|c} A & A & \cdots & A \end{array}\right]$. That is, $\mathrm{per}(A_\alpha)$ is nonzero for some α, if and only if $\mathrm{perrank}\left(\begin{array}{c|c|c|c} A & A & \cdots & A \end{array}\right) = n$. But Conjecture 2.1.8 asserts that it is enough for perrank of $\left[\begin{array}{c|c} A & A \end{array}\right]$ to be n. In other words, if A is an AJT-matrix then the coefficient of some x^α is zero, for which, each $\alpha_i \leq 2$. That clearly shows that Conjecture 2.1.8 implies AJT Conjecture.

Yang Yu [23] also generalized Conjecture 2.1.8 as:

Conjecture 2.1.10. *If* A, B *are invertible* $n \times n$ *matrices over the field* $\mathbb{F}$*, then there is an* $n \times n$ *sub-matrix* C *of the* $n \times 2n$ *matrix* $\left[\begin{array}{c|c} A & B \end{array}\right]$ *so that* $\mathrm{perrank}(C) = n$*. In other words,* $\mathrm{perrank}\left(\begin{array}{c|c} A & B \end{array}\right) = n$.

Over the field $GF(3)$ Conjecture 2.1.10 is a consequence of the Alon-Tarsi Basis Conjecture, and it also implies the weak 3-flow Conjecture [4, 17, 19].

2.2 Characterization of real matrices with small perrank

Now that the perrank is defined and the importance of studying it is clear, we are interested in better understanding it. One of the ways to serve this purpose is characterizing matrices according to their perrank. We begin by describing some transformations that preserve the perrank of matrices.

Proposition 2.2.1. *For each matrix* A, $\mathrm{perrank}(A) = \mathrm{perrank}(A^T)$*, where* A^T *is the transpose of* A.

Proof: $\mathrm{per}(B) = \mathrm{per}(B^T)$ for each square matrix B. Also, B is a sub-matrix of A if and only if B^T is a sub-matrix of A^T. So a square sub-matrix of A has nonzero permanent if and only if its transpose has a nonzero permanent. □

Proposition 2.2.2. *The* per *is invariant under permutation of rows and columns.*

Proof: Permuting rows changes the order of the summation in the definition of the permanent, and permuting columns reorders the product in that definition. So the permanent remains untouched. □

Corollary 2.2.3. *The* perrank *is invariant under permutation of rows and columns.*

Proposition 2.2.4. *Scaling a row or column of a matrix by a nonzero number does not change the* perrank *of the matrix.*

Proof: Scaling a row or column multiplies the permanent of the matrix by that scalar. So a sub-matrix of the original matrix has nonzero permanent if and only if the same sub-matrix of the scaled matrix has a nonzero permanent.

Definition 2.2.5. *Two matrices are* ***equivalent*** *if they can be obtained from each other by permuting rows and columns, and taking transposes, or by scaling rows and columns by nonzero elements to provide* 1*'s.*

Proposition 2.2.6. *Appending or removing zero rows or columns does not change the* perrank *of matrix.*

If an entry of a matrix is nonzero, one can scale the row or column containing that entry in order to make that entry equal to 1. We often scale rows and columns to get 1's in the first row and the first column of the matrix. Using the transformations described above simplifies the classification of matrices of specific perrank.

Definition 2.2.7. *A matrix is said to be* ***reduced*** *if it is obtained from a matrix by deleting all zero rows and zero columns.*

For example, the reduced form of

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 3 & 6 & 12 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 6 & 12 \end{bmatrix}.$$

Definition 2.2.8. *The **term rank** of a matrix is the maximum number of nonzero entries with no two in the same row or column. Equivalently, it is the minimum number of rows and columns that cover all the nonzero entries of the matrix. (See Theorem 1.2.1 of [7])*

Definition 2.2.9. *A set of n entries of an $n \times n$ matrix with no two in the same row or column is called a diagonal of the matrix.*

Diagonals of an $n \times n$ matrix are in a one to one correspondence with the set of permutations on n objects.

Definition 2.2.10. *The product of all entries on a given diagonal of a matrix is called a diagonal product of the matrix.*

Proposition 2.2.11. *For any matrix A,* $\text{perrank}(A) \leq \text{termrank}(A)$.

Proof: If $\text{termrank}(A) = k$, then each diagonal of each $(k+1) \times (k+1)$ sub-matrix of A has a zero. Thus each diagonal product of that sub-matrix is zero. That is, each term in the permanent is zero, and consequently the permanent is zero. Hence $\text{perrank}(A) \leq k$. □

Proposition 2.2.12. *The matrix A has* perrank *zero if and only if $A = O$.*

Proof: If A is a zero matrix, then each sub-matrix of A of any order will have permanent zero, so we say the permanent rank of A is zero. Conversely, if A has at least one nonzero entry, say a_{ij}, then the permanent of the 1×1 sub-matrix $[a_{ij}]$ is nonzero, hence $\text{perrank}(A) \geq 1$. □

Theorem 2.2.13. *The real reduced matrix A is of* perrank *1, if and only if it is equivalent to a matrix of form:*

$$j_n^T = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}, \text{ or } k_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Proof: Clearly j_n, j_n^T and k_2 have perrank 1. We determine when reduced $m \times n$ matrices, with $m \leq n$, have perrank 1 for $m = 1, 2$, and then show for $m \geq 3$ there are not any reduced perrank 1 matrices.

Let A be a reduced $1 \times n$ matrix with at least one nonzero entry. It cannot have a zero entry, since a zero entry means a zero column. So, $A = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$.

Now consider the case that $m = 2$. Since $\text{perrank}(A) = 1$ and A is reduced, A has no 0 entries, otherwise by scaling the first row and column and permuting we may assume that A is equivalent to $\begin{bmatrix} 0 & 1 \\ 1 & \star \end{bmatrix}$, which has perrank 2. So, We may assume that A has the form $A = \begin{bmatrix} 1 & 1 & \cdots \\ 1 & a_{22} & \cdots \end{bmatrix}$, $a_{ij} \neq 0$. In order to have $\text{perrank}(A) = 1, \text{per}(A[\{1,2\};\{1,2\}]) = a_{22} + 1 = 0$, so $a_{22} = -1$. If $n \geq 3$, then $a_{3\,3} = -1$, but then $\text{per}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \neq 0$. Thus $n = 2$, hence $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ up to equivalence, and clearly $\text{perrank}(k_2) = 1$.

Let A be a reduced $3 \times n$ matrix with perrank 1, and $n \geq 3$. Every 2×2 sub-matrix of A should also be of perrank at most one. It cannot be zero, because after permutation and scaling, the matrix would be: $A = \begin{bmatrix} \star & 1 & \star & \cdots \\ 1 & 0 & 0 & \cdots \\ \star & 0 & 0 & \cdots \end{bmatrix}$, and then the upper left 2×2 sub-matrix will have perrank two. So, every 2×2 sub-matrix should have perrank exactly equal to one. That is, up to equivalence they are $k_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, specifically $A[\{1,2\};\{1,2\}]$ and $A[\{1,2\};\{1,3\}]$. Thus $A = \begin{bmatrix} 1 & 1 & 1 & \cdots \\ 1 & -1 & -1 & \cdots \\ \star & \star & \star & \cdots \end{bmatrix}$. But then $\text{perrank}(A[\{1,2\};\{2,3\}]) = 2$. So there is no $3 \times n$ matrix $(n \geq 2)$ with perrank 1. □

Here we introduce a technique, using algebraic geometry. In order to characterize matrices with higher perrank, say k, we look at the all $(k+1) \times (k+1)$ sub-matrices of a generic $m \times n$ matrix, and their permanents equal to zero.

Definition 2.2.14. *Let $\mathbb{F}$ be a field. $\mathbb{F}[x_1, x_2, \ldots, x_n]$ is the ring of polynomials in n variables $x_1, \ldots, x_n$, with coefficients in $\mathbb{F}$.*

For example, let

$$A = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix},$$

where x_{ij} are the variables. Then the permanent of each $(k+1) \times (k+1)$ sub-matrix is a multi-variable polynomial in $\mathbb{F}[x_{11}, \ldots, x_{mn}]$, where $\mathbb{F}$ is a field. Using Gröbner bases, we are able to solve some of these systems of polynomial equations and decide if there is a matrix with the assumed perrank and size, and if there is, if we can characterize them. For example, let $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ be a generic matrix over $\mathbb{F}[a, b, c, d, e, f]$. Then the ideal generated by the permanents of the 2×2 sub-matrices of A is $I = \langle ae + bd, af + cd, bf + ce \rangle$. A Gröbner basis for I is $B = \{ae + bd, af + cd, bcd, bf + ce, cde\}$. This means perrank$(A) \leq 1$ if and only if $ae + bd = 0, af + cd = 0, bcd = 0, bf + ce = 0$, and $cde = 0$ simultaneously. In particular, this calculation shows that if a 2×3 matrix has perrank ≤ 1, then it must have a zero entry. Without loss of generality, we may take $c = 0$. Then per$(A) = 0$ if and only if $ae + bd = 0, af = 0$, and $bf = 0$. That is, either $f = 0$ or $a = b = 0$. This results in having a zero column or a zero row, respectively.

Furthermore, in order to simplify the computations, if some x_{1j} or x_{i1} is assumed to be nonzero, it is set to be 1 in the generic matrix. For instance, for a 2×3 matrix with no zero entries, it is equivalent to $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e & f \end{bmatrix}$. Therefore, the ideal generated by the permanents of the 2×2 sub-matrices of A is $I = \langle e + 1, f + 1, e + f \rangle$. And its Gröbner basis is $B = \{1\}$. That means that I is the whole ring. Consequently, the polynomials in I do not have any common zeros; specifically, the generators of I do not have any common zeros. That is not all the permanents vanish simultaneously. Hence the permanent of at least of the 2×2 sub-matrices is nonzero. As a result, perrank$(A) = 2$.

An immediate corollary of Hilbert's weak Nullstellensatz [5] asserts that if $\mathbb{F}$ is an algebraically closed field, then an ideal I in $\mathbb{F}[x_1, x_2, \ldots, x_n]$ contains 1. That is, I is the whole ring if and only if the polynomials in I do not have any common zeros in $\mathbb{F}^n$. In order to force some variables, say $x_1, \ldots, x_r$, to be nonzero, we also add a polynomial of the form $x_1 \cdot \ldots \cdot x_r \cdot z - 1$ to the ideal, for a dummy variable z. For example, the above example can be looked as a matrix with no zero entries over $\mathbb{F}[a, b, c, d, e, f]$, where a, b, c, d, e, f are variables, and with the ideal $I = \langle ae + bd, af + cd, bf + ce, abcdefz - 1\rangle$. Then the Gröbner basis is $\{1\}$.

R. C. Laubenbacher and I. Swanson in their paper "Permanental Ideals" [11] define the ideals generated by the permanents of $k \times k$ sub-matrices of a generic matrix, and call them *permanental ideals*. Furthermore they give a primary decomposition of ideals generated by the 2×2 sub-permanents of a generic matrix.

Before proving the next theorem which characterizes the matrices with perrank 2, we need some more tools.

Remark 2.2.15. *It is easy to see that if B and C are square matrices, then*

$$\operatorname{per}\begin{bmatrix} A & B \\ C & O \end{bmatrix} = \operatorname{per}(B)\operatorname{per}(C).$$

Lemma 2.2.16.

$$\operatorname{perrank}\begin{bmatrix} A & B \\ C & O \end{bmatrix} \geq \operatorname{perrank}(B) + \operatorname{perrank}(C).$$

Proof: Note that here B and C are not necessarily square matrices. Suppose $B'_{r\times r}$ and $C'_{s\times s}$ are sub-matrices of B and C, respectively, with nonzero permanents. Let $A'_{r\times s}$ be the sub-matrix of A corresponding to the rows of B' and columns of C'. Then

$$\operatorname{per}\begin{bmatrix} A' & B' \\ C' & O \end{bmatrix}_{(r+s)\times(r+s)} = \operatorname{per}(B')\operatorname{per}(C')$$

is nonzero, by Remark 2.2.15. Thus the perrank of the original matrix is at least $r + s$. □

Lemma 2.2.17. *Let A be a reduced $m \times n$ matrix with $3 \leq m \leq n$. Assume that* $\operatorname{perrank}(A) = 2$*, and that it has no $r \times s$ zero sub-matrix with $r + s = m$. Then A does not have more than* *one zero in any row and column.*

Proof: Suppose to the contrary that A has two zeros in one row, so without loss of generality

$$A = \left[\begin{array}{cc|c} 0 & 0 & 1 \\ \hline & & \\ & & \end{array}\right].$$

Then the perrank of the lower left block must be 1. So by Theorem 2.2.13 without loss of generality

$$A = \left[\begin{array}{cc|c} \mathbf{0} & \mathbf{0} & 1 \\ \hline 1 & 1 & \\ \mathbf{0} & \mathbf{0} & \\ \vdots & \vdots & \\ \mathbf{0} & \mathbf{0} & \end{array}\right], A = \left[\begin{array}{cc|c} 0 & \mathbf{0} & 1 \\ \hline 1 & \mathbf{0} & \\ \vdots & \vdots & \\ 1 & \mathbf{0} & \\ 0 & \mathbf{0} & \\ \vdots & \vdots & \\ 0 & \mathbf{0} & \end{array}\right], \text{ or } A = \left[\begin{array}{cc|c} \mathbf{0} & \mathbf{0} & 1 \\ \hline 1 & 1 & \\ 1 & -1 & \\ \mathbf{0} & \mathbf{0} & \\ \vdots & \vdots & \\ \mathbf{0} & \mathbf{0} & \end{array}\right].$$

In the first case there is an $(m-1) \times 2$ zero sub-matrix; in the second case there is a zero column; and in the third case there is an $(m-2) \times 2$ zero sub-matrix. Similarly, A cannot have two zeros in a column, since $m \leq n$. □

Theorem 2.2.18. *The real reduced matrix $A_{m \times n}$ with $m \leq n$ has* perrank 2 *if and only if one of the following occurs:*

1. *A is equivalent to $\begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix}$, with $a \neq -1$.*
2. *A is a $2 \times n$ matrix with $n \geq 3$.*
3. *A has an $r \times s$ zero sub-matrix such that $r + s = m$, and A up to equivalence is of one of these forms:*

(a)

$$\left[\begin{array}{c|ccc} \star & 1 & \cdots & 1 \\ \hline 1 & & & \\ \vdots & & O & \\ 1 & & & \end{array}\right]$$

(b)

$$\left[\begin{array}{rr|ccc} a & b & 1 & \cdots & 1 \\ \hline 1 & 1 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \end{array}\right]_{3\times n} ; a, b \in \mathbb{R}$$

(c)

$$\left[\begin{array}{rr|rr} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

4. *$3 \leq m \leq n$ and A has no $r \times s$ zero sub-matrix such that $r + s = m$, and is equivalent to a matrix of the following forms:*

   (a) *A 3×3 matrix with $\mathrm{per}(A) = 0$.*

   (b) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & a & b & c \\ 1 & -a & -b & -c \end{bmatrix}; abc \neq 0, ab + ac + bc = 0.$

   (c) *A matrix of the form* $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & a & b & c \\ 1 & d & e & f \end{bmatrix}$, *with a, b, c, d, e, f nonzero and described as* *below:*

$$a = -\frac{(d+1)f^2 + (2e + d^2 + de - d + 1)f + de + d^2 - e}{f(d+1+e-f)+1},$$

$$b = -\frac{df^2 + (2e + d^2 + de)f + de + d^2 + d}{(f+1)(d+1+e-f)},$$

$$c = \frac{f(d-1+e+f)}{d+1+e-f},$$

$$d = \frac{-e^2f-e^2-e+f+1-ef^2+\sqrt{(e+f)(e+1)(f+1)((f+1)e^2+(f^2-6f+1)e+f^2+f)}}{2(ef+e+f)},$$

$$\text{or } d = \frac{e^2f+e^2-4\,ef-3\,e-3\,f+ef^2+f^2+\sqrt{(e+f)(e+1)(f+1)((f+1)e^2+(f^2-6f+1)e+f^2+f)}}{2(ef+e+f)}.$$

Proof: Let A be a reduced m by n real matrix with perrank$(A) = 2$. Then clearly $m, n \geq 2$.

Case 1: Suppose that $m = 2$.

If A has no zero entries, then without loss of generality, A has the form

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & a_2 & \cdots & a_n \end{bmatrix}.$$

Since perrank$(A) > 1$, the characterization of perrank 1 matrices (Theorem 2.2.13) implies that either $n \geq 3$ or $n = 2$ and $a_2 \neq -1$, as desired.

Case 2: $m, n \geq 3$.

Case 2a: A has at least two 0's in the same row. Then we may assume that A has the form

$$\left[\begin{array}{c|c} 0 \quad 0 & 1 \cdots 1\; 0 \cdots 0 \\ \hline B & C \end{array}\right].$$

By the perrank 1 characterization, up to equivalence, either B has a zero column, B has all but one row that is 0's, or B is equivalent to

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The first case does not occur, since A is reduced. Suppose the second case occurs. Then up to equivalence

$$A = \left[\begin{array}{cc|c|c} 0 & 0 & 1\cdots 1 & 0\cdots 0 \\ \hline 1 & 1 & u^T & v^T \\ \hline O & O & W & X \end{array}\right].$$

Since $\operatorname{perrank}(A) \leq 3$, $X = O$, and A has just one 1 in the first row. It follows that A is equivalent to a matrix of the form

$$\left[\begin{array}{c|c} * & 1\cdots 1 \\ \hline 1 & \\ \vdots & O \\ 1 & \end{array}\right].$$

Suppose the third case occurs. Then A is equivalent to a matrix of the form

$$\left[\begin{array}{cc|c} 1 & 1 & \multirow{2}{*}{U} \\ 1 & -1 & \\ \hline \multicolumn{2}{c|}{O} & V \end{array}\right].$$

Then $\operatorname{perrank}(V) = 1$, V either has term rank 1 or is equivalent to

$$A = \left[\begin{array}{cc|cc} 1 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{W}} \\ 1 & -1 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{O}} & 1 & 1 \\ & & 1 & -1 \end{array}\right].$$

If the latter happens, let W be

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then the permanents of the 3×3 sub-matrices of A must be zero. Hence

$$\begin{aligned} \operatorname{per}(A(4;2)) &= \quad a + b + c + d = 0, \\ \operatorname{per}(A(4;1)) &= -a - b + c + d = 0, \\ \operatorname{per}(A(3;2)) &= -a + b - c + d = 0, \\ \operatorname{per}(A(3;1)) &= \quad a - b - c + d = 0. \end{aligned}$$

Solving this system of 4 linear equations in 4 variables we get $a = b = c = d = 0$, thus $W = O$.

If the former happens, then either V is a column of ones, or a row of ones. In the case of a column of ones the matrix is equivalent to

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & -1 & b \\ \hline 0 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{array}\right],$$

with a, b arbitrary.

In the case of a row of all ones, we show that if V has at least three 1's, then W should be zero. Let

$$A = \left[\begin{array}{cc|cccc} 1 & 1 & a & b & c & \cdots \\ 1 & -1 & d & e & f & \cdots \\ \hline 0 & 0 & 1 & 1 & 1 & \cdots \end{array}\right].$$

By solving the equations perrank$(A(\ ;\{1,j\}))$, for $j = 3, 4, 5$ we get $b = -a, c = -a$, and $c = -b$. But that happens only if $a = b = c = 0$. Thus $W = O$. If V has at most two 1's, then A is equivalent to

$$\left[\begin{array}{cc|cc} 1 & 1 & a & -a \\ 1 & -1 & b & -b \\ \hline 0 & 0 & 1 & 1 \end{array}\right];\ a, b \text{ arbitrary.}$$

Case 2b: $m, n \geq 3$, A has a 0, and no row or column of A has two 0s. Let $m = n = 3$, then it is a 3×3 matrix with per$(A) = 0$. So, its perrank is at most 2, and by the characterization of matrices with permanent rank 1, Theorem 2.2.13, perrank(A) is exactly 2.

Now, suppose it is a 3×4 matrix. It cannot have more than 3 zeros, each of which lie in a different row and column, by Lemma 2.2.17. Let I be the ideal generated by the polynomials from taking permanents of all the 3×3 sub-matrices of a generic 3×4 matrix A reduced as below:

$$A = \begin{bmatrix} x_1 & 1 & 1 & 1 \\ 1 & x_2 & a & b \\ 1 & c & x_3 & d \end{bmatrix}.$$

If A has 3 zeros, without loss of generality we can set $x_1 = x_2 = x_3 = 0$. Then $I = \langle ac + ad + bc, a + b + d, b + c + d, a + c, abcdz - 1\rangle$. But then the Gröbner basis of I is 1. That means that I is the whole ring, and by the corollary of the weak Nullstellensatz, these polynomials do not have any common zeros. So, construction of such a matrix is impossible. In other words, if A is a 3×4 matrix with 3 zeros, no two in a row or column, then $\operatorname{perrank}(A) > 2$.

Similarly, when it has two zeros, the ideal I generated by the permanents of all the 3×3 sub-matrices is the whole ring. For a 3×4 matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & a & b & c \\ 1 & d & e & f \end{bmatrix}$$

with 1 zero the ideal is:

$$I = \langle ae + af + bd + bf + cd + ce, b + c + e + f, a + c + d + f, a + b + d + e\rangle,$$

which has a Gröbner basis

$$B = \{a + d, b + e, c + f, de + df + ef\}.$$

The first three basis elements imply that $d = -a, e = -b, f = -c$. But then the last basis element indicates to have $ab + ac + bc = 0$. So all the 3×4 matrices with only one zero and perrank 2 are of the form:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & a & b & c \\ 1 & -a & -b & -c \end{bmatrix}; a, b, c \neq 0, ab + ac + bc = 0.$$

This family of matrices can then be described by two parameters.

Case 2c: The same approach has been chosen in order to characterize the 3×4 matrices with no zero entries, but the Gröbner basis for the ideal of these matrices is too big (with 132 polynomials). We were not able to solve the system of polynomial equations directly, but using a shift in the variables we solved the equations and found a plausible solution. Define the new variables to be

$$\begin{array}{lll} a' = a+1, & b' = b+1, & c' = c+1, \\ d' = d+1, & e' = e+1, & f' = f+1. \end{array}$$

Then the ideal is:

$$I = \langle a'e' + b'd' - 2, a'f' + c'd' - 2, b'f' + c'e' - 2, a' + b' + c' + d' + e' + f' - 6\rangle.$$

Solving this system of polynomial equations, all the parameters can be described in terms of two parameters e and f:

$$a = -\frac{(d+1)f^2 + (2e + d^2 + de - d + 1)f + de + d^2 - e}{f(d+1+e-f)+1},$$

$$b = -\frac{df^2 + (2e + d^2 + de)f + de + d^2 + d}{(f+1)(d+1+e-f)},$$

$$c = \frac{f(d-1+e+f)}{d+1+e-f},$$

$$d = \frac{-e^2f-e^2-e+f+1-ef^2+\sqrt{(e+f)(e+1)(f+1)((f+1)e^2+(f^2-6f+1)e+f^2+f)}}{2(ef+e+f)},$$

$$\text{or } d = \frac{e^2f+e^2-4ef-3e-3f+ef^2+f^2+\sqrt{(e+f)(e+1)(f+1)((f+1)e^2+(f^2-6f+1)e+f^2+f)}}{2(ef+e+f)}.$$

Now, let A be a 3×5 matrix. It must contain some zero entry, since the Gröbner basis of the ideal generated by the permanents of its 3×3 sub-matrices is the whole ring. That is, it includes 1. So, suppose it has one zero entry:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & a & b & c & d \\ 1 & e & f & g & h \end{bmatrix}.$$

Then the ideal I has a Gröbner basis

$$B = \{a+e, b+f, c+g, d+h, ef-gh, eg-fh, eh+fh+gh, f^2h, fg+fh+gh, fh^2, g^2h, gh^2\}.$$

Solving the system of polynomial equations obtained from this basis implies

$$a = 0, b = 0, c = 0, d = -h, e = 0, f = 0, g = 0.$$

So, A will have some 2×3 zero sub-matrix, which does not satisfy the assumed conditions. This means that every 3×5, and hence any larger matrix with no $r \times s$ zero sub-matrix with $r + s = 3$ has perrank > 2.

The remaining case is a 4×4 matrix. Let

$$A = \begin{bmatrix} x & 1 & 1 & 1 \\ 1 & a & b & c \\ 1 & d & e & f \\ 1 & g & h & k \end{bmatrix}$$

The Gröbner basis of the ideal generated by the permanents of all the 3×3 sub-matrices of A is $B = \{a, b, c, d, e, f, g, h, k\}$, which means that it has a 3×3 zero sub-matrix. So, such a 4×4 matrix also does not exist. □

2.3 Relation between termrank and perrank

In this section we purpose a conjecture, relating the term rank and the permanent rank of matrices. Theorem 2.1.2 asserts that for any matrix A,

$$\text{perrank}(A) \geq \frac{\text{rank}(A)}{2}.$$

Yang Yu [23] mentions that the inequality is tight for the family of matrices k_{2n}, since $\text{rank}(k_{2n}) = 2n$, and $\text{perrank}(k_{2n}) = n$. Note that he asserts up to equivalence these are the only *known* matrices for which the perrank is exactly half of the rank. On the other hand this instance of such matrices led us to a conjecture:

Conjecture 2.3.1 (Bryan Shader). *For any matrix A,*

$$\text{perrank}(A) \geq \left\lceil \frac{\text{termrank}(A)}{2} \right\rceil,$$

and for even termrank *the equality holds if and only if up to equivalence and reduction*

$$A = \bigoplus \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

It can easily be seen that theorem 2.1.2 implies that Conjecture 2.3.1 is true for invertible matrices. Studying this conjecture suggests that maybe the perrank of a matrix, unlike the rank, is more related to the structure of the matrix, say the zero pattern, rather than the individual entries of it.

2.3.1 Validity of the Conjecture for term ranks less than 4:

This section studies the validity of the Conjecture 2.3.1 for small term ranks. The validity of the conjecture for termrank less than four is clear from the characterizations in Theorems 2.2.13 and 2.2.18.

Furthermore, we can look at the families of matrices which satisfy the equality in the conjecture. Using the following lemmas we are able to extend a matrix that satisfies the equality in Conjecture 2.3.1 to a family of matrices for which the inequality is sharp.

Lemma 2.3.2. *Let A be an $m \times n$ matrix, and x be a $1 \times n$ vector. Assume* $\operatorname{perrank}(A) = k$. *Then* $\operatorname{perrank}\left(\begin{array}{c} A \\ \hline x \end{array}\right) \leq k+1$.

Proof: Since $\operatorname{perrank}(A) = k$, permanent of each $(k+1) \times (k+1)$ sub-matrix of A is zero. We show that the permanent of any $(k+2) \times (k+2)$ sub-matrix of $\left[\begin{array}{c} A \\ \hline x \end{array}\right]$ is zero, which is equivalent to saying that its perrank is at most $k+1$. If the sub-matrix does not include any entries of x, then it is a sub-matrix of A, and by assumption its permanent is zero. Suppose it includes some entries of x. By Laplace expansion [7] along the last row, the permanent of this $(k+2) \times (k+2)$ sub-matrix is a linear combination of permanents of $(k+1) \times (k+1)$ sub-matrices of A. By assumption, the permanents of all of those $(k+1) \times (k+1)$ sub-matrices are zero, hence the permanent of the $(k+2) \times (k+2)$ is zero. Hence $\operatorname{perrank}\left(\begin{array}{c} A \\ \hline x \end{array}\right) \leq k+1$. □

Note that this lemma tells us that, like the rank of the matrix, adjoining a vector to a matrix can increase its perrank by at most one.

Lemma 2.3.3. *Let A be an $m \times n$ matrix and x be a vector. Then*

$$\operatorname{perrank}\left(\begin{array}{c|c} A & O \\ \hline 1 & x \end{array}\right) = \operatorname{perrank}(A) + 1.$$

Proof: Clearly $\operatorname{perrank}\left(\begin{array}{c|c} A & O \end{array}\right) = \operatorname{perrank}(A)$. In one hand,

$$\operatorname{perrank}\left(\begin{array}{c|c} A & O \\ \hline x & 1 \end{array}\right) \geq \operatorname{perrank}(A) + 1,$$

because if B is a maximal sub-matrix of A with nonzero permanent, then $\left[\begin{array}{c|c} B & O \\ \hline x' & 1 \end{array}\right]$, x' the sub-matrix of x corresponding to the columns of B, is a maximal sub-matrix of the original matrix with nonzero permanent, using Laplace expansion on the last column. On the other hand, by Lemma 2.3.2

$$\operatorname{perrank}\left(\begin{array}{c|c} A & O \\ \hline x & 1 \end{array}\right) \leq \operatorname{perrank}\left(\begin{array}{c|c} A & O \end{array}\right) + 1 = \operatorname{perrank}(A) + 1.$$

□

Lemma 2.3.4. *Let A be an $n \times n$ matrix, with odd $n \geq 4$, such that for each i, up to equivalence $A(i,i)$ has the zero pattern:*

$$A(i,i) = \begin{bmatrix} J_2 & & O \\ & \ddots & \\ O & & J_2 \end{bmatrix},$$

where J_2's are 2×2 matrices of all ones on the main diagonal of the matrix, zeros everywhere else. Because $A(1,1)$ has the above form, then A should be of this form, where $\#$'s denote a deleted entry:

$$A = \begin{bmatrix} \# & \# & \# & & & \\ \# & 1 & 1 & & & \\ \# & 1 & 1 & & & \\ & & & 1 & 1 & \\ & & & 1 & 1 & \\ & & & & & \ddots \end{bmatrix}.$$

Then considering $A(2,2)$ one can say that A should be of the following form:

$$A = \begin{bmatrix} 1 & \# & 1 & & & \\ \# & \# & \# & & & \\ 1 & \# & 1 & & & \\ & & & 1 & 1 & \\ & & & 1 & 1 & \\ & & & & & \ddots \end{bmatrix}.$$

And similarly looking at $A(3,3)$:

$$A = \begin{bmatrix} 1 & 1 & \# & & & \\ 1 & 1 & \# & & & \\ \# & \# & \# & \# & & \\ & & \# & 1 & 1 & \\ & & & 1 & 1 & \\ & & & & & \ddots \end{bmatrix}.$$

But then considering all these three cases, $A(4,4)$ will have the zero pattern:

$$A = \begin{bmatrix} 1 & 1 & 1 & & & \\ 1 & 1 & 1 & & & \\ 1 & 1 & 1 & & & \\ & & & 1 & 1 & \\ & & & 1 & 1 & \\ & & & & & \ddots \end{bmatrix},$$

which contradicts the assumption. □

Now that we have all the tools let's see how we can extend a matrix which satisfies the equality in the conjecture 2.3.1 to a family of sharp matrices. One of these matrices is

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

which is a specific case of the Theorem 2.2.18, part 3.b. Let $x = (x_1, x_2, 1)$, with x_1, x_2 arbitrary. Then

$$A' = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1 & 0 \\ \hline x_1 & x_2 & 1 \end{array}\right].$$

We have termrank$(A) = 3$ and perrank$(A) = 2$, which means they satisfy the equality

$$\text{perrank}(A) = \left\lceil \frac{\text{termrank}(A)}{2} \right\rceil.$$

Similarly, if $A = \bigoplus_{n-times} k_2$, and $x = (x_1, x_2, \ldots, x_{2n}, 1)$, with x_i arbitrary; we have termrank$(A) = 2n$ and perrank$(A) = n$. Furthermore, for

$$A' = \left[\begin{array}{cccc|c} 1 & 1 & & & \\ & & & O & 0 \\ 1 & -1 & & & \\ & & \ddots & & \vdots \\ & & & 1 & 1 \\ O & & & & 0 \\ & & & 1 & -1 \\ \hline x_1 & & \cdots & x_{2n} & 1 \end{array}\right],$$

termrank$(A') = 2n + 1$ and perrank$(A') = n + 1$, which satisfy the equality. The above process suggests an inductive procedure that we were not able to complete, but here are the steps:

If termrank$(A) = 1$, then it has at least one nonzero element, and that implies perrank$(A) \geq 1$. The cases $t = 2, 3$ were also checked above.

Let termrank$(A) = t \geq 4$, and suppose the conjecture is true for all the matrices with termrank at most $t - 1$.

Definition 2.3.5. *A collection of e rows and f columns of $A_{m\times n}$, say E and F respectively, such that $A(E; F) = O$ is called a covering of rows and columns of A. If $e + f$ is as small as possible, then it is called a minimum covering. If $e, f > 0$ it is said to be a mixed covering. If $f = 0$, it is a row covering, and if $e = 0$, then it's called a column covering. Clearly if A has no zero row or columns and $m < n$, then there is not a minimum column covering.*

Theorem 1.2.1 of [7] asserts that if there is a minimum covering of e rows and f columns for a matrix, then termrank$(A) = e + f$.

Let $A_{m\times n}$, with $m \leq n$, have a minimum mixed covering of rows and columns.

$$A = \left[\begin{array}{c|c} \star & B_{e\times(n-f)} \\ \hline C_{(m-e)\times f} & O \end{array}\right]; e + f = t \text{ and } e, f \neq 0.$$

Since this is a minimum covering, then termrank(A) $= t$, and termrank(B) $= e$ and termrank(C) $= f$. By induction hypothesis perrank(B) $\geq \left\lceil \frac{e}{2} \right\rceil$ and perrank(C) $\geq \left\lceil \frac{f}{2} \right\rceil$. Finally, by Lemma 2.2.16 perrank(A) $\geq \left\lceil \frac{e}{2} \right\rceil + \left\lceil \frac{f}{2} \right\rceil \geq \left\lceil \frac{e+f}{2} \right\rceil = \left\lceil \frac{t}{2} \right\rceil$.

Now suppose that A does not have a minimum mixed covering, that is it has a minimum row covering, and assume it is not a square matrix. Then it has t independent nonzero elements. Permute rows and columns such that they are placed in (i, i) entries of the matrix. If one of the columns which does not contain these elements is omitted, the matrix still will have a minimum covering of t rows. Deleting all such columns we will end up with a $t \times t$ square matrix. So, if we prove that this square matrix satisfies the inequality, it also proves that the original rectangular matrix satisfies the inequality.

Here we consider two cases, t is even and t is odd. First assume that t is odd, so $t = 2k + 1$ for some positive integer k. Then by induction hypothesis perrank($A(t; t)$) $\geq k$, and perrank($A(t, t)$) $= k$ if and only if

$$A(t,t) = \bigoplus k_2 = \begin{bmatrix} 1 & 1 & & 0 & 0 \\ -1 & 1 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & 1 & 1 \\ 0 & 0 & & -1 & 1 \end{bmatrix}.$$

But that should be true for any $A(i; i)$, which is impossible for $t \geq 3$, because of Lemma 2.3.4 below. So, perrank($A(t; t)$) $\geq k + 1$, that is perrank(A) $\geq k + 1$.

Now consider that case that t is even, that is $t = 2k + 2$ for some positive integer k. Every matrix of termrank $2k + 2$ has a sub-matrix of termrank $2k + 1$, which means by induction hypothesis perrank of that sub-matrix is at least $\left\lceil \frac{2k+1}{2} \right\rceil = k + 1$. That is, perrank($A$) $\geq k + 1 = \left\lceil \frac{t}{2} \right\rceil$.

The only remaining step that we were **not** yet able to complete is to prove the case of equality for matrices with even termrank. We have done this for termrank 2 in 2.2.13, and for termrank 4 in 2.2.18.

These theorems and lemmas can be used as tools for further work on characterizing matrices with a fixed permanent rank, and also to study the conjectures mentioned in this chapter. We hope that we can extend these results in the future.

Glossary

S_n the symmetric group of degree n, that is the set of all the permutations of the n symbols. 1, 30

#**P** **problem** The class of functions that can be computed by counting Turing Machines of polynomial time complexity [21]. 2

AJT-matrix A matrix A over a field $\mathbb{F}$ such that there exists a vector x over $\mathbb{F}$ such that both x and Ax are nowhere-zero [1]. 9

bipartite graph A graph whose vertices can be divided into two disjoint sets V_1 and V_2 such that every edge connects a vertex in V_1 to one in V_2; that is, V_1 and V_2 are independent sets. 1

cycle A closed path with no repeated vertices. 31

cycle cover A set of disjoint cycles which are subgraphs of a graph and contain all the vertices of the graph. 1

determinant Let $A = (a_{ij})$ be an $n \times n$ matrix over any commutative ring, the determinant of A, written $\det(A)$, is defined by

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)},$$

Where S_n is the symmetric group of order n. 1, 2, 32

diagonal Let $A = (a_{ij})$ be an $m \times n$ matrix over any commutative ring, $m \leq n$, for any $\sigma \in S_n$, the sequence $(a_{1\sigma(1)}, \ldots, a_{m\sigma(m)})$ is called a diagonal of A. 11, 25

diagonal product Let $A = (a_{ij})$ be an $m \times n$ matrix over any commutative ring, $m \leq n$, for any $\sigma \in S_n$, the product $\prod_{i=1}^{m} a_{i\sigma(i)}$ is called a diagonal product of A. 11

directed graph A graph whose edges are ordered pairs of the vertices. 1

graph An ordered pair $G = (V, E)$ of a set V of vertices and a set E of edges, which are 2-element subsets of V. 30, 31

Hamiltonian cycle A cycle through a graph that visits each vertex exactly once. 2

matching A set of edges of a graph without common vertices. 31

nowhere-zero A vector with no zero entries. 2, 3, 6, 30

path A sequence of vertices of a graph such that from each of its vertices there is an edge to the next vertex in the sequence. 30

perfect matching A matching which saturates all the vertices of the graph. 1

permanent Let $A = (a_{ij})$ be an $m \times n$ matrix over any commutative ring, $m \leq n$, the permanent of A, written per(A), is defined by

$$\text{per}(A) = \sum_{\sigma} \prod_{i=1}^{m} a_{i\sigma(i)},$$

where the summation extends over all one-to-one functions from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$. The permanent of A is the sum of all diagonal products of A. For $n \leq m$ the permanent of A is defined to be equal to the permanent of A^T [13]. 1, 2, 4, 31

permanent rank the size of a largest square sub-matrix of the matrix with nonzero permanent. 1, 2, 4, 22

rank The maximum number of linearly independent column vectors of the matrix, which is equal to the maximum number of linearly independent row vectors of it. Equivalently, it is the size of a largest square sub-matrix of the matrix with nonzero determinant. 2

sub-matrix A matrix which is obtained by omitting some rows and/or columns of the original matrix. There are two ways to denote such a matrix. Let $A_{m\times n}$ be a matrix, and $\alpha \subset \{1, 2, \ldots, m\}, \beta \subset \{1, 2, \ldots, n\}$. Then $A[\alpha; \beta]$ is the matrix obtained by keeping the rows of α and columns of β, and deleting the other rows and columns. Also, $A(\alpha; \beta)$ is the matrix obtained by omitting the rows of α and columns of β, and keeping the rest of rows and columns, which is the complementary matrix of $A[\alpha; \beta]$. 2, 4

term rank Let $A_{m\times n}$ be a matrix. The term rank of A is the maximum number of the nonzero entries of A no two in the same row or column. Equivalently, it is the minimum number of the lines (rows and columns) in A that contain all of the nonzero entries in A. 4, 11, 22

transpose If $A_{m\times n} = [a_{ij}]$, then transpose of A is defined to be $A^T_{n\times m} := [a_{ji}]$. 9, 10

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