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ON THE EXISTENCE OF NOWHERE-ZERO VECTORS FOR LINEAR TRANSFORMATIONS

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Abstract

A matrix A over a field F is said to be an *AJT matrix* if there exists a vector x over F such that both x and Ax have no zero component. The *Alon–Jaeger–Tarsi (AJT) conjecture* states that if F is a finite field, with $|F| \geq 4$, and A is an element of $\mathrm{GL}_n(F)$, then A is an AJT matrix. In this paper we prove that every nonzero matrix over a field F, with $|F| \geq 3$, is similar to an AJT matrix. Let $AJT_n(q)$ denote the set of $n \times n$, invertible, AJT matrices over a field with q elements. It is shown that the following are equivalent for $q \geq 3$: (i) $AJT_n(q) = \mathrm{GL}_n(q)$; (ii) every $2n \times n$ matrix of the form $(A|B)^t$ has a nowhere-zero vector in its image, where A, B are $n \times n$, invertible, upper and lower triangular matrices, respectively; and (iii) $AJT_n(q)$ forms a semigroup.

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1. Introduction

A matrix A over a field F is said to be an *AJT matrix* if there exists a vector x over F such that both x and Ax are nowhere-zero vectors (that is, each component of them is nonzero). The *Alon–Jaeger–Tarsi conjecture* (AJT conjecture) states that if F is a finite field, with $|F| \geq 4$, and A is an element of $\mathrm{GL}_n(F)$, then A is an AJT matrix. In [2] the conjecture was proved for $|F| = p^k$, where p is a prime number and $k \geq 2$ is an integer. In [5] it was shown that the conjecture is true for $|F| \geq n \geq 4$.

Our main result is that every nonzero matrix over a field F, with $|F| \geq 3$, is similar to an AJT matrix. We also provide necessary and sufficient conditions for a matrix to be an AJT matrix. Throughout this paper, $M_{m,n}(F)$ denotes the set of all $m \times n$ matrices over the field F, and F^n indicates $M_{n,1}(F)$. Also, $\ker(A)$ and $\mathrm{im}(A)$ denote the kernel and the image of the linear transformation corresponding to the matrix A, respectively. A matrix $A = (a_{ij})$ is an *upper Hessenberg matrix* if $a_{ij} = 0$ for $i > j + 1$. In that case, A^t is called a *lower Hessenberg matrix*. An $n \times n$ matrix $C = (c_{ij})$ is a *circulant matrix* if $c_{ij} = c_{i+1,j+1}$, where the subscripts are taken

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modulo n. Let $AJT_n(q)$ denote the set of $n \times n$, invertible, AJT matrices over a field with q elements. A natural question arises here: which classic subgroups of $\mathrm{GL}_n(q)$ are subsets of $AJT_n(q)$? It is easily seen that the set of invertible circulant matrices is a subset of $AJT_n(q)$.

The permanent of an $n \times n$ matrix $A = (a_{ij})$ is defined as

$$\mathrm{Per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

The sum here extends over all elements σ of the symmetric group S_n.

2. Every nonzero square matrix is similar to an AJT matrix

In this section we prove that under similarity the AJT conjecture is true.

THEOREM 1. *Every nonzero matrix $A \in M_n(F)$, with $|F| \geq 3$, is similar to an AJT matrix.*

PROOF. Suppose that A is in its rational canonical form, and without loss of generality assume that its $m \times m$ zero block, if it exists, is located in its upper left corner. Any nonzero block of A has the form

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & b_1 \\ 1 & 0 & \cdots & 0 & b_2 \\ 0 & 1 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & b_{k-1} \\ 0 & 0 & \cdots & 1 & b_k \end{pmatrix}.$$

We consider the following cases.

(1) The last column of B contains a nonzero element, say b_j. Since B is similar to its transpose B^t [4, Section 3.2.3], we can assign a proper coefficient to the jth row of B and add it to the rest of the rows to obtain a nowhere-zero vector.

(2) The last column of B is zero. Then B is similar to

$$C = \begin{pmatrix} 1 & -1 & \cdots & 0 & 0 \\ 1 & -1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

That is, $C = PBP^{-1}$, where P is the matrix that when applied to B from the left replaces the first row of B with the sum of its first and second rows, and leaves the other rows unaltered. It is easily seen that C is an AJT matrix.

Now, since A is assumed to be block diagonal, we can replace all nonzero blocks on the diagonal of A with their similar AJT versions given in (1) and (2) above, and call the matrix thus obtained $\tilde{A}$. Consider a nonzero row of $\tilde{A}$, say the ith row. Let $\tilde{A}_j$ denote the jth row of $\tilde{A}$. Assume that Q is the invertible matrix such that $(Q\tilde{A})_j = \tilde{A}_j + \tilde{A}_i$, for every j, $1 \leq j \leq m$, and $(Q\tilde{A})_k = \tilde{A}_k$, for any k, $m+1 \leq k \leq n$. It is not hard to see that $Q\tilde{A} = Q\tilde{A}Q^{-1}$. Now, since every nonzero block of $\tilde{A}$ is an AJT matrix we conclude that $Q\tilde{A}Q^{-1}$ is an AJT matrix. □

REMARK 2. A similar proof shows that every nonzero matrix $A \in M_n(F)$, with $|F| \geq 5$, is similar to a matrix B with the property that for any u, $v \in F^n$, there exists $x \in F^n$ such that $x - u$ and $Bx - v$ are nowhere-zero vectors.

3. A generalization of AJT matrices

The following theorem was proved in [5]. The proof is rather long. Theorem 3 generalizes this result and provides a short and simple proof for it.

THEOREM. *Suppose that $A \in M_{m,n}(F)$, with $|F| = q$, and $q > m+1$. There is a vector $x \in F^n$ such that neither x nor Ax has any zero entries if and only if no row of A is zero.*

THEOREM 3. *Let $A \in M_{m,n}(F)$, with $|F| > m+1$. Then for any $u \in F^n$ and $v \in F^m$ there exists $x \in F^n$ such that $x - u$ and $Ax - v$ are nowhere-zero vectors if and only if A has no zero row.*

PROOF. One direction is clear. For the other direction, let S be a finite subset of F with at least $m+2$ elements, containing all entries of u. Hence, there are $(|S|-1)^n$ vectors x in S^n such that $x - u$ is a nowhere-zero vector, and since A has no zero row, the product of at most $(|S|-1)^{n-1}$ of these vectors and the ith row of A is equal to the ith entry of v, $1 \leq i \leq m$. Obviously, $(|S|-1)^n > m(|S|-1)^{n-1}$ implies the existence of $x \in F^n$ such that $x - u$ and $Ax - v$ are nowhere-zero vectors. □

REMARK 4. The previous theorem does not hold for $|F| = m+1$. For example, consider the $m \times 2$ matrix

$$B = \begin{pmatrix} f_1 & 1 \\ \vdots & \vdots \\ f_m & 1 \end{pmatrix},$$

where $F = \{0, f_1, \ldots, f_m\}$ and u, v are zero vectors. Then for any nowhere-zero vector $x = (x_1, x_2)^t$, Bx has a zero component, since the equation $x_1 z + x_2 = 0$ in z, takes a nonzero solution in F. For $|F| = m+1$, the mean of the number of zero entries of Ax, say M, is less than or equal to $(mm^{n-1})/m^n = 1$, where the mean is taken over all nowhere-zero vectors x. If the number of nonzero entries in at least one row of A is not equal to 2, then $M < 1$ and A is an AJT matrix. If $M = 1$ and A has at least three nonzero columns, then there exists a nowhere-zero vector x such that Ax has more than one zero. Hence, there exists a nowhere-zero vector y such that Ay has less than one zero, that is, A is an AJT matrix. Hence, if the number of nonzero entries in at least one row of A is not equal to two, or if A has at least three nonzero columns, then A is an AJT matrix over a field F of size $m+1$. Thus, all $m \times n$ matrices with no zero row which are not AJT matrices over a field F of size $m+1$ are obtained from B by adding zero columns to it, permuting, or multiplying its rows by nonzero scalars from F. This too follows from the probabilistic method used in [3, Proof of Theorem 1].

COROLLARY 5. *Let F be an infinite field and $A \in M_{m,n}(F)$. Then for any $u \in F^n$,* $\ker(A)$ *contains a vector x such that $x-u$ is a nowhere-zero vector if and only if the row space of A contains no vector $e_i = (0, 0, \ldots, 1, 0, \ldots, 0)$, where the ith component is* 1.

PROOF. One direction is obvious. For the other direction, note that the row space of A has no e_i if and only if the reduced row echelon matrix of A, say R, has no vector e_i as one of its rows. Let R_f be the submatrix of R obtained from the columns corresponding to the free variables of $Rx = 0$ with the possible zero rows removed. Now, according to Theorem 3, there exist x_f and y_f such that $x_f - u_f$ and $y_f - (-u_p)$ are nowhere-zero vectors and $R_f x_f = y_f$, where u_f, u_p is the partitioning of u into components corresponding to the free and pivot variables of $Rx = 0$, respectively. It suffices to take $-y_f$ for the pivot variables of $Rx = 0$, and this determines a vector x in the null space of R with the desired property. □

REMARK 6. The proof of Corollary 5 gives a necessary and sufficient condition for the kernel of a matrix to contain a nowhere-zero vector over an arbitrary field: $\ker(A)$ contains a nowhere-zero vector if and only if R_f is an AJT matrix.

Now, we state the following trivial but useful lemma.

LEMMA 7. *Given $u, v \in F^n$ and a triangular matrix $A \in \mathrm{GL}_n(F)$, with $|F| \geq 3$, there exists $x \in F^n$ such that $x-u$ and $Ax-v$ are nowhere-zero vectors.*

PROOF. Since $\mathrm{Per}(A) = \det(A) \neq 0$, we can apply [2, Proposition 2]. □

REMARK 8. Clearly, for every permutation matrix P and Q, A is an AJT matrix if and only if PAQ is an AJT matrix. More generally, for any $u, v \in F^n$, there exists $x \in F^n$ such that $x-u$ and $Ax-v$ are nowhere-zero vectors if and only if, for any $u, v \in F^n$, there exists $y \in F^n$ such that $y-u$ and $PAQy-v$ are nowhere-zero vectors. So, using Lemma 7, we can find other families of invertible AJT matrices by permuting rows and columns.

Let us generalize Lemma 7 in the following theorem which immediately implies that every upper or lower Hessenberg matrix $H \in \mathrm{GL}_n(F)$, with $|F| \geq 4$, is an AJT matrix.

THEOREM 9. *Let $A = (a_{ij})$ be a matrix in $\mathrm{GL}_n(F)$, with $|F| \geq 4$, such that $a_{ij} = 0$ for $i > j+2$ (or similarly $a_{ij} = 0$ for $j > i+2$). Then, given $u, v \in F^n$, there exists $x \in F^n$ such that $x-u$ and $Ax-v$ are nowhere-zero vectors.*

PROOF. The two cases $|F| = 4$ and $n < 4$ follow from [2, Proposition 1] and Theorem 3, respectively. So, we may suppose that $|F| \geq 5$ and $n \geq 4$. According to Remark 8, we may rearrange the rows of A to obtain a matrix R such that for each k, $1 \leq k \leq n-1$, the nonzero leading entry of the $(k+1)$th row of R is in the same column as the nonzero leading entry of its kth row or in a column to the right of it and prove the theorem for R. Note that $r_{i,i-2} = r_{i,i-1} = 0$ implies that $r_{ii} \neq 0$. Otherwise,

$$\det(R) = \det\begin{pmatrix} B & C \\ 0 & D \end{pmatrix} = 0,$$

where B is an $(i-1) \times (i-1)$ matrix, and D is an $(n-i+1) \times (n-i+1)$ matrix whose first column is zero, contradicting our hypothesis that A is invertible. Thus, each column of R contains at most three nonzero leading entries. This fact, together with $|F| \geq 5$, enables us to make a vector $x = (x_1, \ldots, x_n)^t$ such that $x - u$ and $Rx - v$ are nowhere-zero vectors by assigning a proper value to x_k and finding proper values for x_{k-1} and x_{k-2}, where $k = n, n-1, \ldots, 3$. □

Our next two theorems show how the problem of the existence of a nowhere-zero vector in the image of a mapping is related to the problem of determining whether a given matrix is an AJT matrix.

THEOREM 10. *Suppose that $A \in M_{m,n}(F)$ has no zero row and* $\operatorname{rank}(A) = r < m$. *Without loss of generality, assume that the first r rows of A are linearly independent, and $A_i = b_{i-r,1}A_1 + \cdots + b_{i-r,r}A_r$, $i = r+1, \ldots, m$, where A_k denotes the kth row of A. Then* $\operatorname{im}(A)$ *contains a nowhere-zero vector if and only if $B = (b_{ij})_{r+1 \leq i \leq m, 1 \leq j \leq r}$ is an AJT matrix.*

PROOF. Clearly, B has no zero row. Assume that $\operatorname{im}(A)$ contains a nowhere-zero vector, that is, there exists $x \in F^n$ such that Ax is a nowhere-zero vector. Let $z = (A_1x, \ldots, A_rx)^t$. Then Bz is a nowhere-zero vector, and therefore B is an AJT matrix. Now, suppose that B is an AJT matrix, that is, there exists $y \in F^r$ such that y and By are nowhere-zero vectors. Let $A = (C|D)^t$ be a partitioning of A into $C \in M_{r,n}(F)$ and $D \in M_{m-r,n}(F)$. Then $\tau_C : F^n \to F^r$, the linear operator corresponding to C, is surjective. Therefore, there exists $x \in F^n$ such that $\tau_C(x) = y$. Clearly, Dx and therefore Ax are nowhere-zero vectors too. □

COROLLARY 11. *Suppose that $A \in M_{m,n}(F)$ has no zero row and that* $\operatorname{rank}(A) = r$. *If $|F| > m - r + 1$, then* $\operatorname{im}(A)$ *contains a nowhere-zero vector.*

PROOF. Apply Theorem 3 to the matrix B in the above theorem. □

REMARK 12. Suppose that $A \in M_{m,n}(F)$ and $\operatorname{rank}(A) = m$. Clearly, $\operatorname{im}(A)$ contains a nowhere-zero vector. Moreover, if $F = GF(p^\alpha)$, $\alpha > 1$, then according to [2] A is an AJT matrix, since it can be extended to an invertible matrix by adding $n - m$ rows to it.

It is well known that any matrix A has a PLU decomposition [4], that is, there exist a lower triangular matrix L, an upper triangular matrix U, one of which is invertible, and a permutation matrix P, such that $A = PLU$. Hence, according to Remark 8, we may restrict our attention to LU decomposable matrices only.

THEOREM 13. *The following are equivalent for* $q \geq 3$.

(1) $AJT_n(q) = \mathrm{GL}_n(q)$.

(2) *Every* $2n \times n$ *matrix of the form* $(A|B)^t$ *has a nowhere-zero vector in its image, where* A, B *are* $n \times n$, *invertible, upper and lower triangular matrices, respectively.*

(3) $AJT_n(q)$ *is closed under multiplication of matrices, that is, it forms a semigroup.*

PROOF. (1) $\Rightarrow$ (2). Let $M = BA^{-1}$. By assumption, there are nowhere-zero vectors x, y such that $Mx = y$. Now, if $z = A^{-1}x$, then $(A|B)^t z = (x|y)^t$.

(2) $\Rightarrow$ (1). Let $M \in \mathrm{GL}_n(q)$. There exists a permutation matrix P such that $PM = LU$, where L and U are lower and upper triangular matrices, respectively. By considering the matrix $(U^{-1}|L)^t$ and using the assumption, we are done.

On the other hand, (1) $\Leftrightarrow$ (3), because of Lemma 7 and the PLU factorization of matrices. □

COROLLARY 14. *Let* $A = LU$ *be an* LU *decomposition for* $A \in \mathrm{GL}_n(F)$, *with* $|F| \geq 4$, *such that the last column of* U^{-1} *and the first column of* L *are nowhere-zero vectors. Then* A *is an AJT matrix.*

PROOF. Set $z = (1, 0, \ldots, 0, c)^t$ in the proof of Theorem 13 for a proper $c \in F$. □

4. Nowhere-zero vectors in the kernel or the image of linear transformations

In this section we provide some criteria for the existence of nowhere-zero vectors in the null space and the image of a linear transformation.

THEOREM 15. *Let* $A \in M_{m,n}(F)$ *be a matrix with no zero row and with at most* k *nonzero entries in each column. If* $|F| > k + 1$, *then* A *is an AJT matrix, and if* $|F| = k + 1$, *then* $\mathrm{im}(A)$ *contains a nowhere-zero vector.*

PROOF. Without loss of generality, assume that A has no zero columns. The proof is by induction on n. For $n = 1$ the assertion is obvious. Suppose that the statement holds for all such A with less than n columns, $n > 1$. Let $\tilde{A}$ be the matrix obtained by omitting the last column of A with its possible zero rows removed. By the induction hypothesis, there exists an $x \in F^{n-1}$ such that $\tilde{A}x$ has the desired property. It is not hard to choose $a \in F$ such that Ay has the same property as $\tilde{A}$, where $y = (x|a)^t$. □

REMARK 16. In [1] it is shown that every (0, 1) matrix with at most two ones in each of its columns and no zero row is an AJT matrix over F, for $|F| \geq 3$.

THEOREM 17. *Let* $A \in M_{m,n}(F)$ *be a* (0, 1) *matrix with at most three ones in each of its columns and no zero row. Then* $\mathrm{im}(A)$ *contains a nowhere-zero vector over* F, $|F| \geq 3$.

PROOF. We apply induction on n. For $n = 1$ the assertion is obvious. Let $n > 1$ and let $\tilde{A}$ be the matrix obtained from omitting a column of A. Now, we consider the following two cases.

(1) $\tilde{A}$ has no zero row. Then, by the induction hypothesis, $\tilde{A}x$ is a nowhere-zero vector for some $x \in F^{n-1}$. Hence, if we assume without loss of generality that the last column of A is removed, then $A(x|0)^t$ will be a nowhere-zero vector.

(2) $\tilde{A}$ has at least one zero row, for every choice of the columns of A. Then, by a permutation of the rows, A will be in the form $(I_n|B)^t$, where B is a matrix with at most two ones in each of its columns, and hence by Remark 16 an AJT matrix. Clearly, A is also an AJT matrix. □

REMARK 18. Let F be a finite field of characteristic 2. Then there exists a $(0, 1)$ matrix with no zero row and $|F| - 1$ ones in each of its columns which is not an AJT matrix over F. Hence, we cannot generalize Remark 16 in this sense. Here, we give an example of such a matrix for $F = GF(4)$:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Clearly, the condition that the nowhere-zero vector x has distinct elements is necessary for Ax to be a nowhere-zero vector. Hence, A is not an AJT matrix over $GF(4)$, since this field has only three nonzero members. Generally, assuming that F is a finite field with $\mathrm{char}(F) = 2$, the same method may be used to construct a matrix with $\binom{|F|}{2}$ rows and $|F|$ columns that is not an AJT matrix over F.

THEOREM 19.

(1) *Suppose that any matrix with at most k nonzero entries in each of its columns and no zero row is an AJT matrix over a field of size $k + 1$. Let A be a matrix with at most $k + 1$ nonzero entries in each of its columns and no zero row. Then* $\mathrm{im}(A)$ *contains a nowhere-zero vector over a field of size $k + 1$.*

(2) *Suppose that for any matrix A with at most l nonzero entries in each of its columns and no zero row over a field of size l,* $\mathrm{im}(A)$ *contains a nowhere-zero vector. Then any matrix B with at most $l - 1$ nonzero entries in each of its columns and no zero row is an AJT matrix over a field of size l.*

PROOF. (1) The proof is similar to that of Theorem 17 and hence omitted.

(2) Suppose that B is an $m \times n$ matrix and define $A = (I_n|B)^t$. Then $\mathrm{im}(A)$ contains a nowhere-zero vector by hypothesis, and hence B is an AJT matrix. □

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