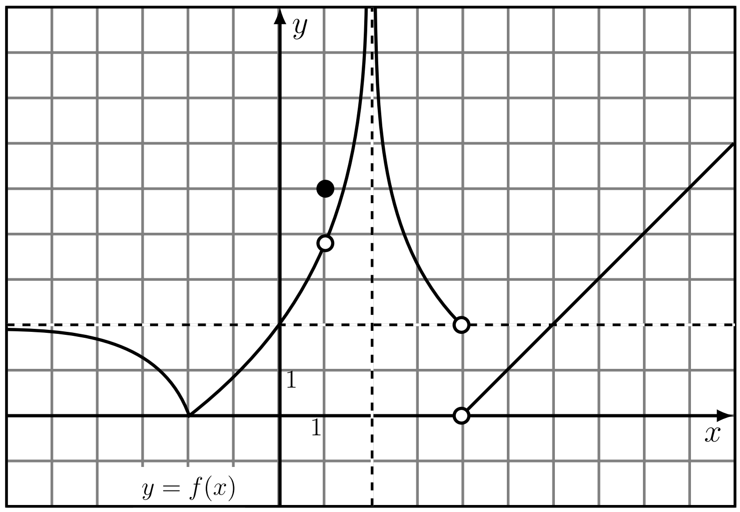
Math 133 - Exam 1

Name: Solution

SHOW ALL OF YOUR WORK.

The graph of the function $y = f(x)$ is shown below.



1. Find the following limits for the function $f(x)$ shown above.

   (a) $\lim_{x\to-\infty} f(x) = 2$

   (b) $\lim_{x\to\infty} f(x) = +\infty$

   (c) $\lim_{x\to 2} f(x) = +\infty$

   (d) $\lim_{x\to 0} \sqrt{f(x)} = \sqrt{2}$

   (e) $\lim_{x\to 4^-} f(x) = 2$

   (f) $\lim_{x\to 4^+} f(x) = 0$

   (g) $\lim_{x\to 1} f(x) \approx 3.8$

   (h) $\lim_{x\to 7} f(x) = 3$

2. List all the discontinuity points of $f(x)$ shown above and list **ALL** reasons why f is discontinuous at that point.

Points (only x)	**Reasons** (Using the graph)	**Reasons** (Using the definition of continuity)
$x = 1$	there is a hole	$\lim_{x\to 1} f(x) \neq f(1)$
$x = 2$	the graph is broken	$f(2)$ is not defined; $\lim_{x\to 2} f(x) = \infty$
$x = 4$	the graph is broken	$f(4)$ is not defined; $\lim_{x\to 4} f(x)$ DNE

3. Evaluate the limit. Show all steps. Mention any theorems used. If the limit does not exist, explain why.

(a) $\underbrace{\lim_{x\to 2} \frac{3x^3-2x^2+x}{x-1}}_{\text{rational function with 2 in its domain} \Rightarrow \text{Cont's at 2}} = \frac{3\cdot 2^3 - 2\cdot 2^2 + 2}{2-1} = 18$

(b) $\lim_{x\to 1} \frac{x^3-x}{x-1} = \lim_{x\to 1} \frac{x\cancel{(x-1)}(x+1)}{\cancel{x-1}} = \lim_{x\to 1} \underbrace{x(x+1)}_{\text{polynomial} \Rightarrow \text{cont's everywhere}} = 1\cdot 2 = 2$

(c) $\lim_{x\to 5} \sqrt{5-x} \rightarrow \begin{cases} \lim_{x\to 5^+} \sqrt{5-x} = 0 \\ \lim_{x\to 5^-} \sqrt{5-x} : \text{DNE} \end{cases} \Rightarrow \lim_{x\to 5} \sqrt{5-x} : \text{DNE}$

(d) $\underbrace{\lim_{x\to\frac{\pi}{2}} \cos^2(x)}_{\cos x \text{ is cont's everywhere, } x^2 \text{ is cont's everywhere} \Rightarrow (\cos x)^2 \text{ is cont's everywhere}} = \cos^2\frac{\pi}{2} = \left(\cos\frac{\pi}{2}\right)^2 = 0$

(e) $\lim_{x\to -\infty} \frac{(3x-1)(2x^2+2)}{-x^3+x^2-x+1} = \lim_{x\to -\infty} \frac{3x\left(1-\frac{1}{3x}\right)2x^2\left(1+\frac{2}{2x^2}\right)}{-x^3\left(1+\frac{x^2}{-x^3}-\frac{x}{-x^3}+\frac{1}{-x^3}\right)} = \lim_{x\to -\infty} \frac{6\cancel{x^3}\left(1-\frac{1}{3x}\right)\left(1+\frac{1}{x^2}\right)}{-\cancel{x^3}\left(1-\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^3}\right)}$

$= \frac{6(1)(1)}{-1(1)} = -6$

4. Find all the points where $f(x) = \ln(\frac{1}{x-1})$ is continuous. Explain your answer.

$\ln(x)$ is cont's on its domain: $(0,\infty)$

and $\frac{1}{x-1} > 0$ when $x > 1$, and is cont's when $x \neq 1$

$\Rightarrow \ln\left(\frac{1}{x-1}\right)$ is cont's on $(1,\infty)$

5. Show that the equation $x^4 + 5x^3 + 5x - 1 = 0$ has at least one real solution in the interval $(-1, 1)$.

Let $f(x) = x^4 + 5x^3 + 5x - 1$

then $f(1) = 10 > 0$

and $f(-1) = -10 < 0$

f is cont's on $[-1,1]$ since it's a polynomial.

By the intermediate value theorem there is a point c in $[-1,1]$ such that $f(c) = 0$, that is, c is a solution of the equation

$$x^4 + 5x^3 + 5x - 1 = 0$$

6. Let $f(x)$ be a function such that $1 - x^2 \leq f(x) \leq \cos x$, for all x in the interval $[-\pi/2, \pi/2]$. Find $\lim_{x\to 0} f(x)$.

$1 - x^2$ is a cont's function everywhere.

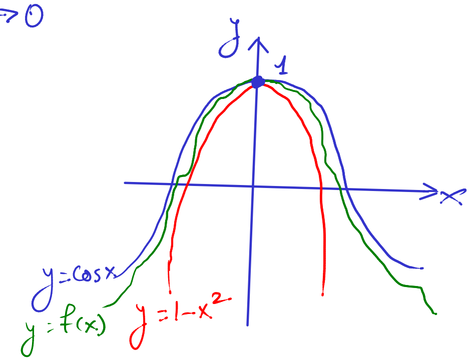
$\cos x$ is a cont's function everywhere.

$\lim_{x \to 0} 1 - x^2 = 1$

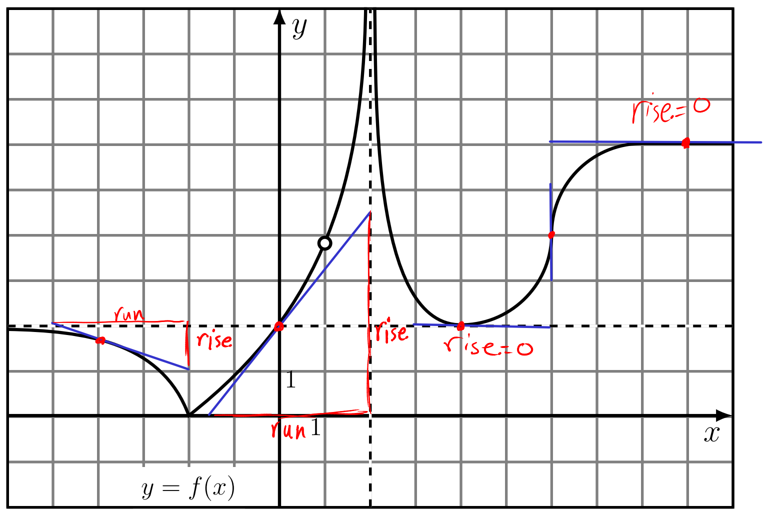
$\lim_{x \to 0} \cos x = 1$

by the squeezing theorem

$\lim_{x \to 0} f(x) = 1$



The graph of the function $y = f(x)$ is shown below.



7. Approximate the following derivatives for the function $f(x)$ shown above. If the derivative does not exist, explain why.

(a) $f'(-4) \approx \frac{-1}{3}$

(b) $f'(-2)$ Undefined, sharp corner

(c) $f'(0) \approx \frac{4.5}{3.5}$

(d) $f'(1)$ undefined, discontinuous

(e) $f'(2)$ Undefined, dis continuous

(f) $f'(4) = 0$

(g) $f'(6)$ undefined, vertical tangent line

(h) $f'(9) = 0$

8. Let $f(x) = 1/x$.

(a) Find $f'(x)$, the derivative function of f.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{\frac{\cancel{x} - (\cancel{x}+h)}{x(x+h)}}{h} = \lim_{h \to 0} \frac{\frac{-h}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{x(x+h)} \cdot \frac{1}{\cancel{h}} = \frac{-1}{x^2}$$

(b) Find $f'(5)$. $f'(5) = \left.\frac{-1}{x^2}\right|_{x=5} = \frac{-1}{5^2} = \frac{-1}{25}$